Lecture7 The Butterworth IIR filter

Lecture 7: The Butterworth IIR filter

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EE3660 Introduction to Digital Signal Processing
National Tsing Hua University

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The analog Butterworth filter (Appendix B.1 of O&S)

This following analog system has a *monotonically decreasing* 嚴格遞減 magnitude response,

$$H(s) = \frac{\Omega_c^N}{\prod_{k=1}^{N}\left(s - \Omega_c e^{j\left(\frac{\pi}{2} - \frac{\pi}{2N}\right)} e^{j\frac{k\pi}{N}}\right)} \quad (1)$$

This is an Nth order Butterworth filter, and its gain at $\Omega = 0$ is $H(j0) = 1$. In fact, its magnitude response is

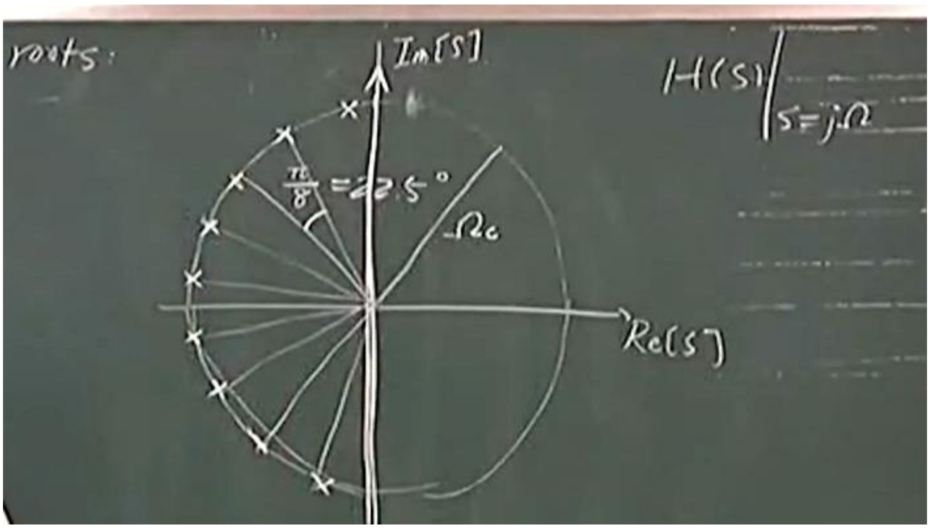
$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}. \quad (2)$$

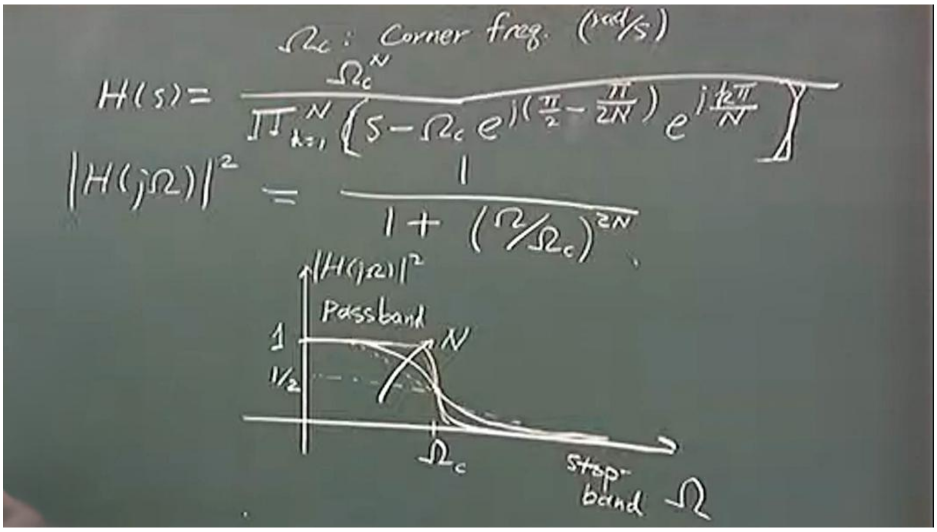
Check:

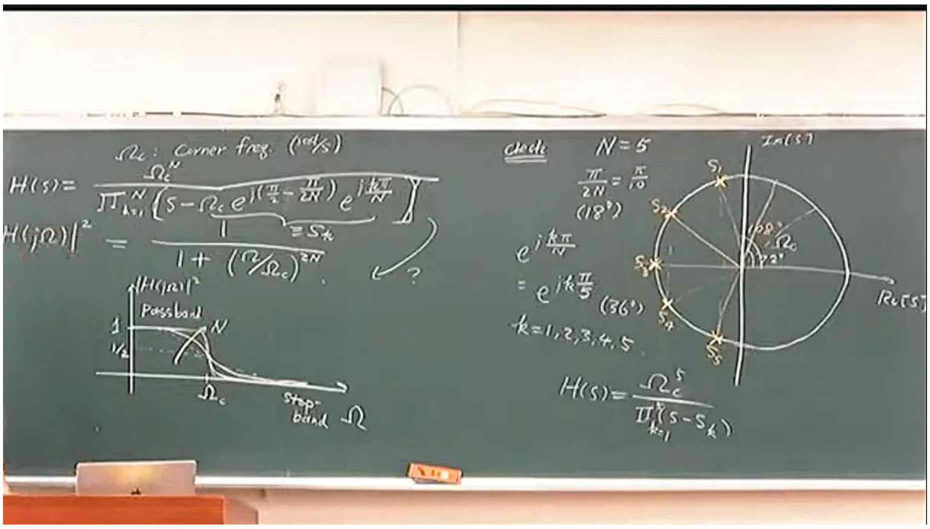
$$H(-s) = \frac{\Omega_c^N}{\prod_{k=1}^{N}\left(-s - \Omega_c e^{j\left(\frac{\pi}{2} - \frac{\pi}{2N}\right)} e^{j\frac{k\pi}{N}}\right)} = \frac{\Omega_c^N}{(-1)^N \prod_{k=1}^{N}\left(s + \Omega_c e^{j\left(\frac{\pi}{2} - \frac{\pi}{2N}\right)} e^{j\frac{k\pi}{N}}\right)}$$

,

$$H(s)H(-s) = \ldots$$


roots:
Im[S]
π/8 = 22.5°
Ωc
Re[S]
H(S)|s=jΩ


Ωc: Corner freq. (rad/s)
H(s) = Ωc^N / Π_{k=1}^{N} {s − Ωc e^{j(π/2 − π/2N)} e^{jkπ/N}}
|H(jΩ)|² = 1 / (1 + (Ω/Ωc)^{2N})
|H(jΩ)|²
Passband
1
1/2
N
Ωc
Stop-band
Ω

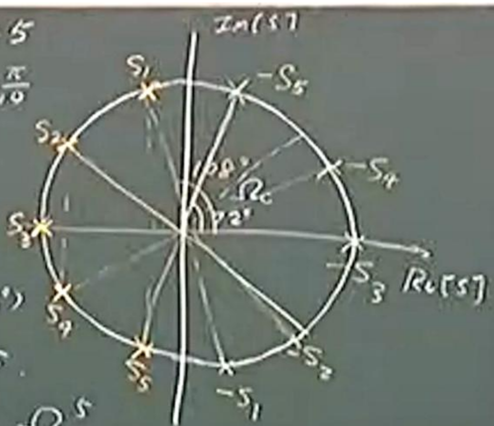

Ωc: Corner freq. (rad/s)
H(s) = Ωc^N / Π_{k=1}^{N} {s − Ωc e^{j(π/2 − π/2N)} e^{jkπ/N}}
≡ Sk
|H(jΩ)|² = 1 / (1 + (Ω/Ωc)^{2N})
Passband
Stop-band
check
N = 5
π/2N = π/10
(18°)
e^{jkπ/N} = e^{jkπ/5} (36°)
k = 1, 2, 3, 4, 5
Im[S]
Re[S]
S1
S2
S3
S4
S5
72°
Ωc
H(s) = Ωc^5 / Π_{k=1}^{5} (s − Sk)

Check $N=5$

$\frac{\pi}{2N} = \frac{\pi}{10}$ (18°)

$e^{j\frac{k\pi}{N}} = e^{jk\frac{\pi}{5}}$ (36°)

$k = 1, 2, 3, 4, 5$



$$H(s) = \frac{\Omega_c^5}{\prod_{k=1}^{5}(s - s_k)}$$

$$H(-s) = \frac{\Omega_c^5}{\prod_{k=1}^{5}(-s - s_k)} = \frac{\Omega_c^5}{(-1)^5 \prod_{k=1}^{5}(s + s_k)}$$

$$\boxed{H(s)H(-s)} = \frac{\Omega_c^{10}}{(-1)^5 \prod_{k=1}^{5}(s - s_k)(s + s_k)} \quad \text{(A)}$$

$$= \frac{\Omega_c^{10}}{(-1)^5 (s^{10} - \Omega_c^{10})}$$

Evaluate (A) at $s = j\Omega$

$$H(j\Omega)H(-j\Omega) = \frac{\Omega_c^{10} \cdot 1}{(-1)^5\left((j\Omega)^{10} - \Omega_c^{10}\right)}$$

$$= \frac{1}{(-1)^5\left((-1) + j^{10}\left(\Omega/\Omega_c\right)^{10}\right)}$$

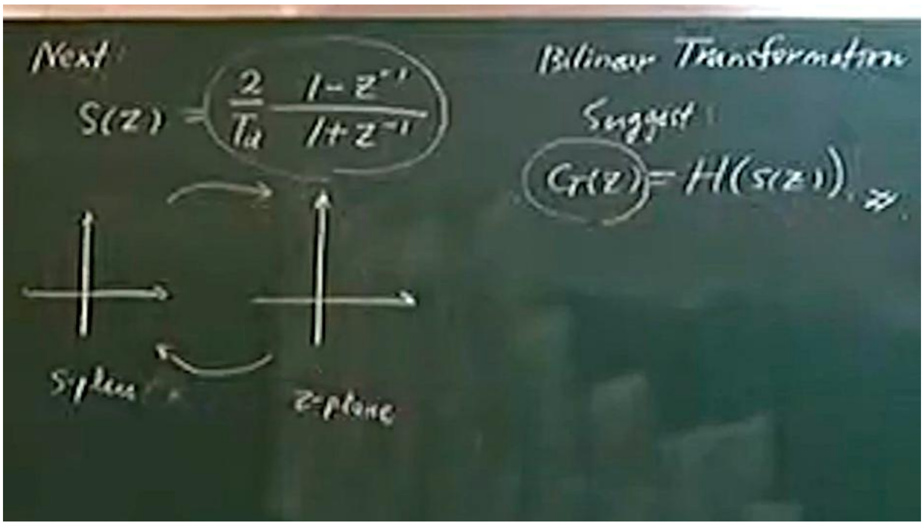
$$= \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{10}}$$

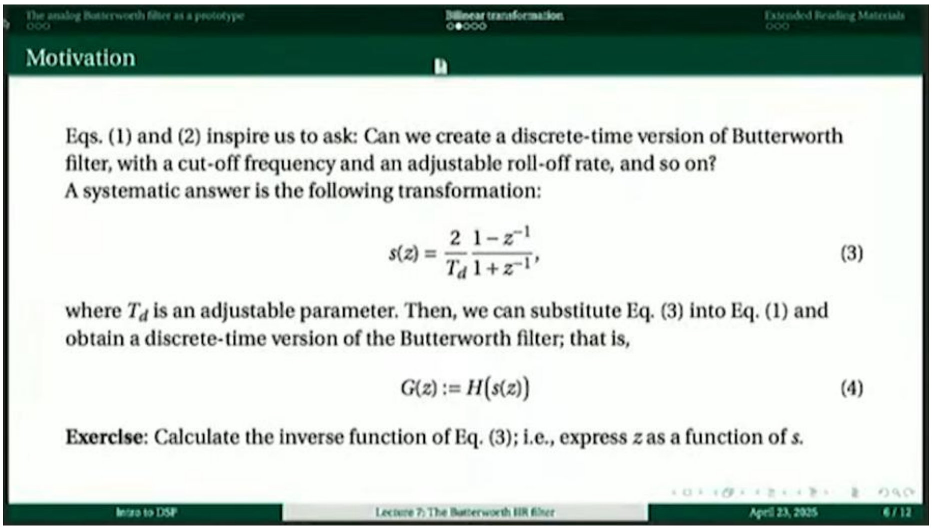
$$= H(j\Omega)H(-j\Omega)$$

$$= H(j\Omega)H^*(j\Omega) = |H(j\Omega)|^2$$

Note: $H(-j\Omega) = H^*(j\Omega)$

Exercise: Generalized to $\forall N \in \mathbb{N}$.


Next
$S(z) = \frac{2}{T_d} \frac{1-z^{-1}}{1+z^{-1}}$
Bilinear Transformation
Suggest
$G(z) = H(s(z))$
s-plane
z-plane


The analog Butterworth filter as a prototype
Bilinear transformation
Extended Reading Materials
Motivation
Eqs. (1) and (2) inspire us to ask: Can we create a discrete-time version of Butterworth filter, with a cut-off frequency and an adjustable roll-off rate, and so on?
A systematic answer is the following transformation:
$$s(z) = \frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}}, \quad (3)$$
where T_d is an adjustable parameter. Then, we can substitute Eq. (3) into Eq. (1) and obtain a discrete-time version of the Butterworth filter; that is,
$$G(z) := H\big(s(z)\big) \quad (4)$$
Exercise: Calculate the inverse function of Eq. (3); i.e., express z as a function of s.
Intro to DSP
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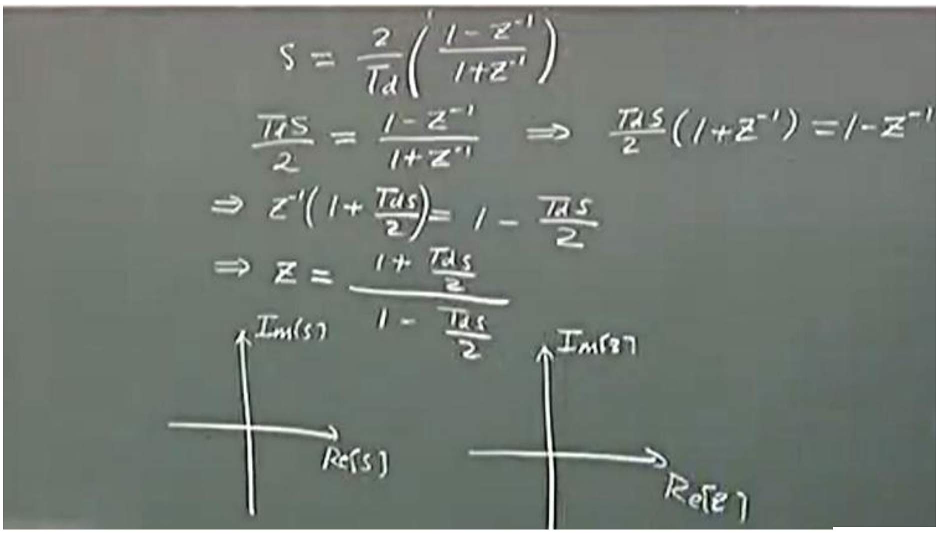
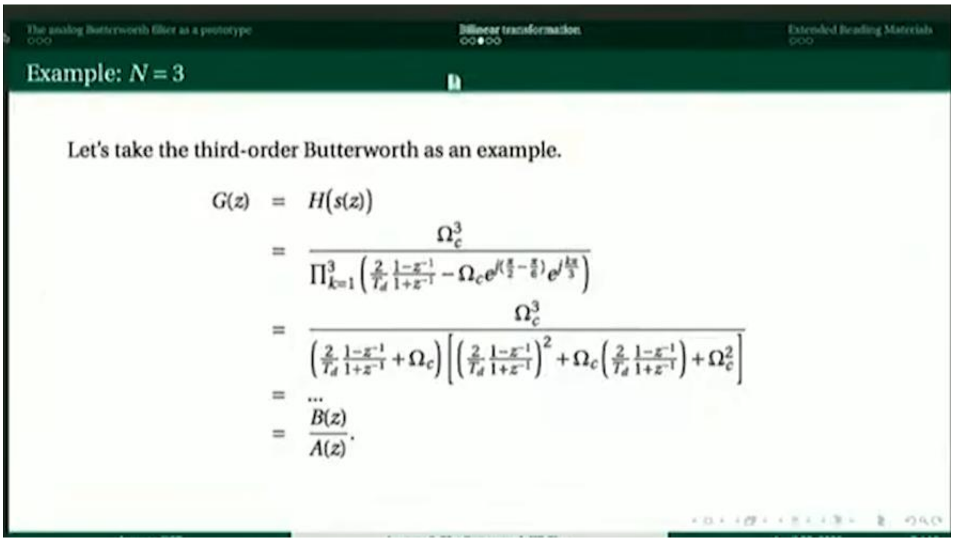
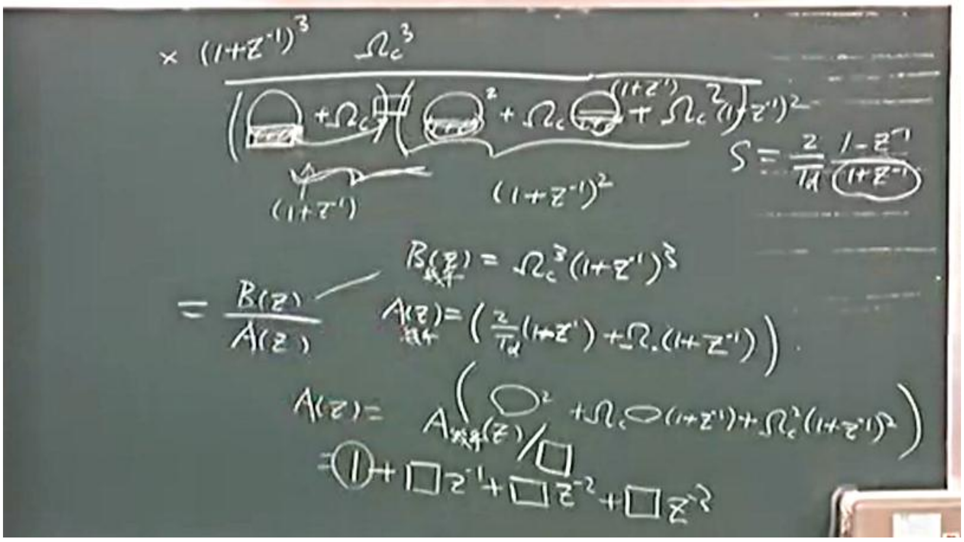
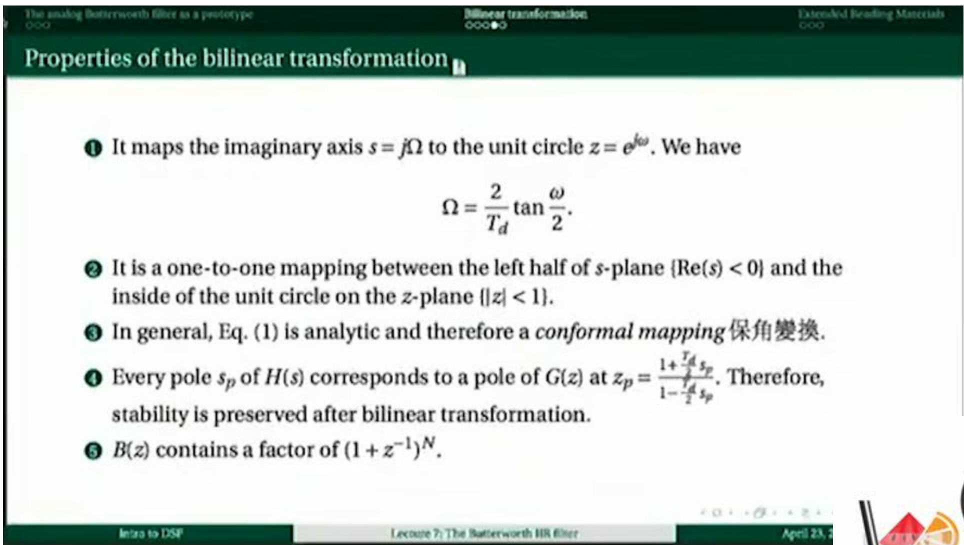

$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$
$\frac{T_d s}{2} = \frac{1-z^{-1}}{1+z^{-1}} \Rightarrow \frac{T_d s}{2}(1+z^{-1}) = 1-z^{-1}$
$\Rightarrow z^{-1}\left(1+\frac{T_d s}{2}\right) = 1-\frac{T_d s}{2}$
$\Rightarrow z = \frac{1+\frac{T_d s}{2}}{1-\frac{T_d s}{2}}$
Im{s}
Re{s}
Im{z}
Re{z}


The analog Butterworth filter as a prototype
Bilinear transformation
Extended Reading Materials
Example: $N = 3$
Let's take the third-order Butterworth as an example.
$$\begin{aligned} G(z) &= H(s(z)) \\ &= \frac{\Omega_c^3}{\prod_{k=1}^{3}\left(\frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}} - \Omega_c e^{j(\frac{\pi}{2}-\frac{\pi}{6})} e^{j\frac{k\pi}{3}}\right)} \\ &= \frac{\Omega_c^3}{\left(\frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}} + \Omega_c\right)\left[\left(\frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \Omega_c\left(\frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}}\right) + \Omega_c^2\right]} \\ &= \dots \\ &= \frac{B(z)}{A(z)}. \end{aligned}$$


$\times\ (1+z^{-1})^3$
$S = \frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}}$
$(1+z^{-1})$
$(1+z^{-1})^2$
$= \frac{B(z)}{A(z)}$
$B(z) = \Omega_c^3 (1+z^{-1})^3$
$A(z) = \left(\frac{2}{T_d}(1-z^{-1}) + \Omega_c(1+z^{-1})\right)$


The analog Butterworth filter as a prototype
Bilinear transformation
Extended Reading Materials
Properties of the bilinear transformation
1. It maps the imaginary axis $s = j\Omega$ to the unit circle $z = e^{j\omega}$. We have
$$\Omega = \frac{2}{T_d}\tan\frac{\omega}{2}.$$
2. It is a one-to-one mapping between the left half of s-plane $\{\mathrm{Re}(s) < 0\}$ and the inside of the unit circle on the z-plane $\{|z| < 1\}$.
3. In general, Eq. (1) is analytic and therefore a *conformal mapping* 保角變換.
4. Every pole s_p of $H(s)$ corresponds to a pole of $G(z)$ at $z_p = \frac{1+\frac{T_d}{2}s_p}{1-\frac{T_d}{2}s_p}$. Therefore, stability is preserved after bilinear transformation.
5. $B(z)$ contains a factor of $(1+z^{-1})^N$.
Intro to DSP
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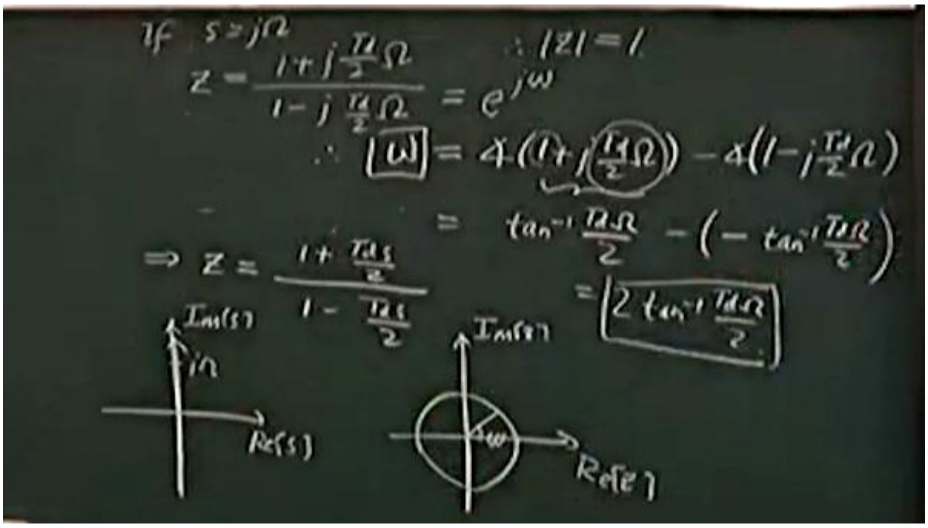
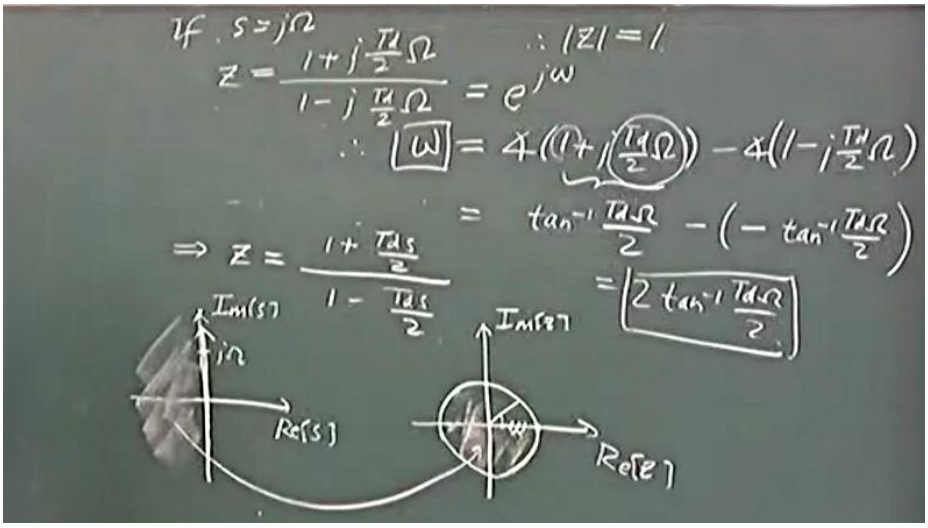
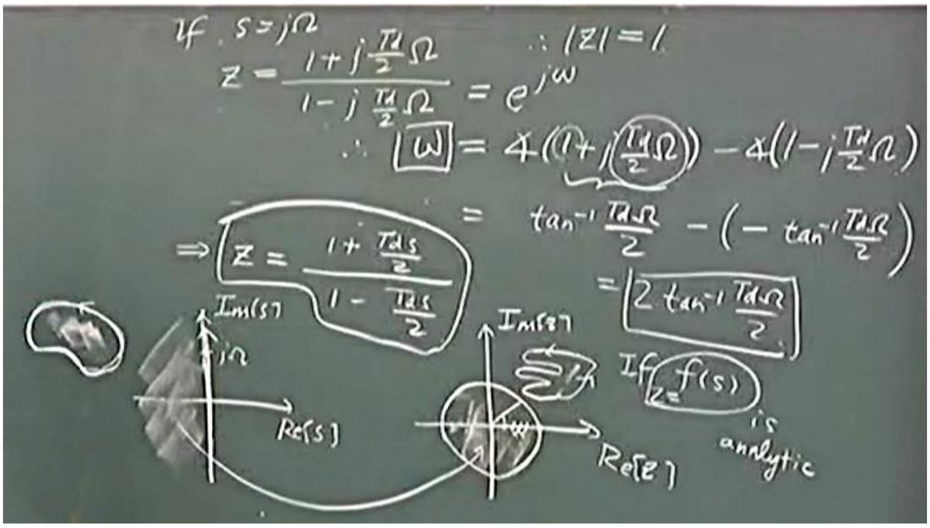
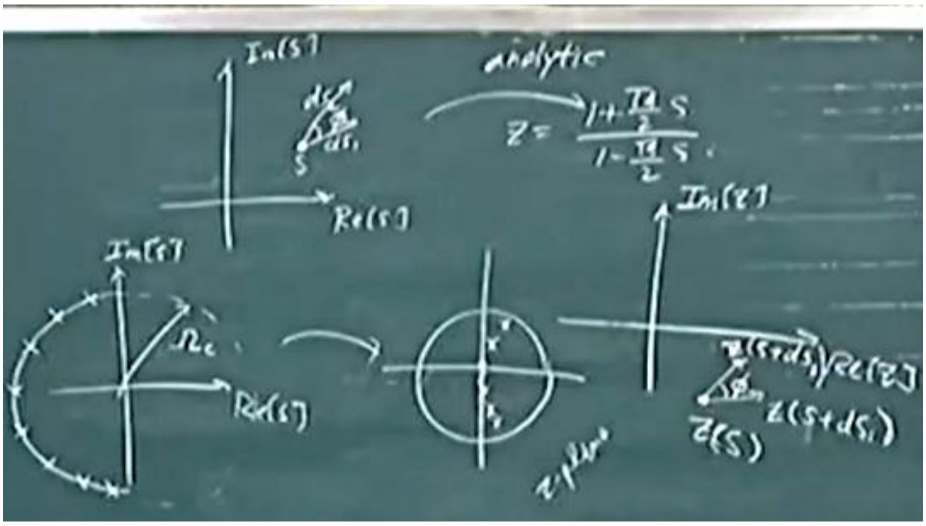
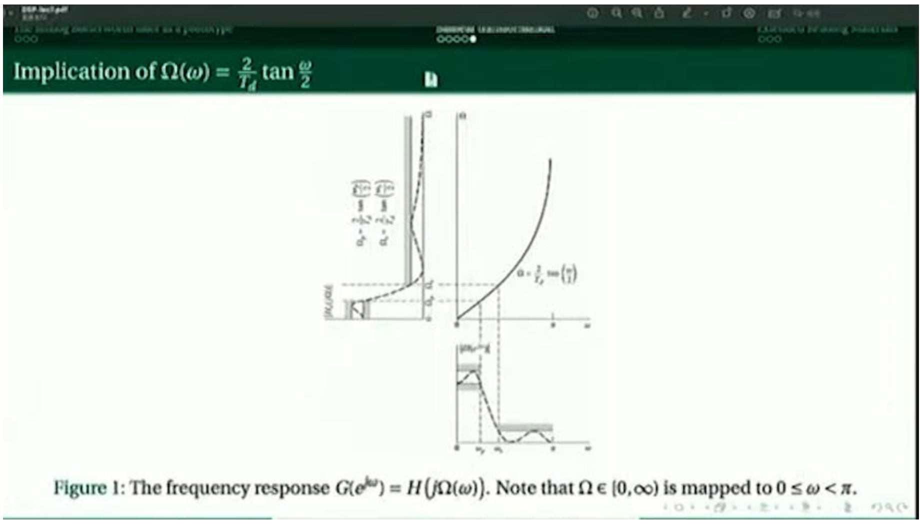
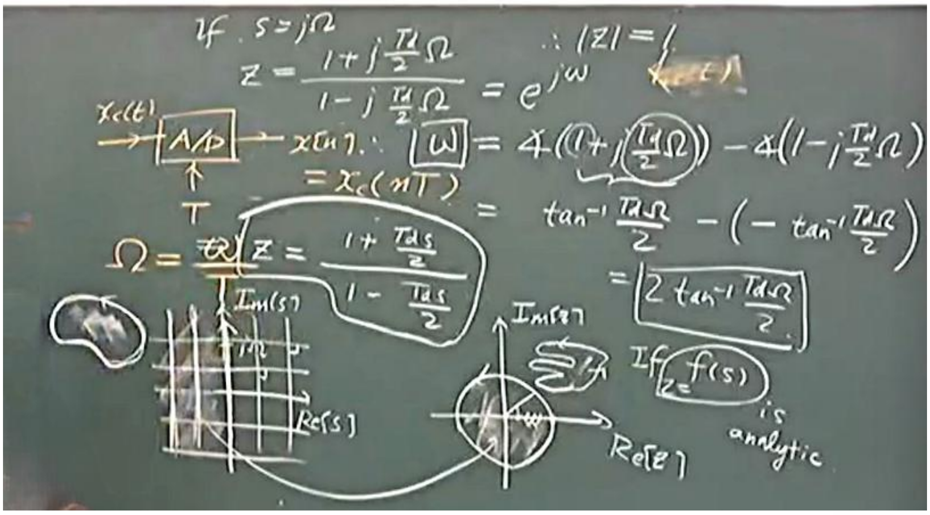

If s = jΩ
∴ |z| = 1
z = (1 + j (Td/2) Ω) / (1 − j (Td/2) Ω) = e^{jω}
∴ ω = ∠(1 + j (Td/2) Ω) − ∠(1 − j (Td/2) Ω)
= tan⁻¹ (TdΩ/2) − (− tan⁻¹ (TdΩ/2))
= 2 tan⁻¹ (TdΩ/2)
⇒ z = (1 + Tds/2) / (1 − Tds/2)
Im[s]
jΩ
Re[s]
Im[z]
ω
Re[z]

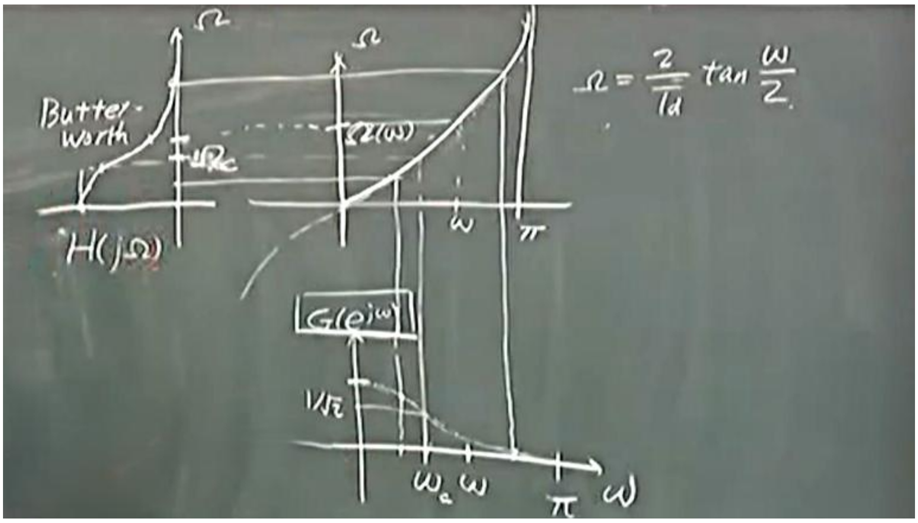

If s = jΩ
∴ |z| = 1
z = (1 + j (Td/2) Ω) / (1 − j (Td/2) Ω) = e^{jω}
∴ ω = ∠(1 + j (Td/2) Ω) − ∠(1 − j (Td/2) Ω)
= tan⁻¹ (TdΩ/2) − (− tan⁻¹ (TdΩ/2))
= 2 tan⁻¹ (TdΩ/2)
⇒ z = (1 + Tds/2) / (1 − Tds/2)
Im[s]
jΩ
Re[s]
Im[z]
ω
Re[z]

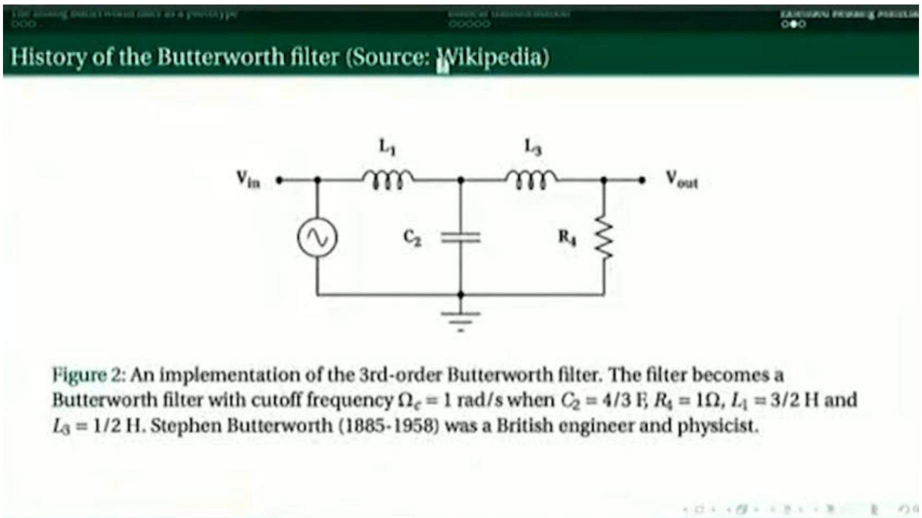

If s = jΩ
∴ |z| = 1
z = (1 + j (Td/2) Ω) / (1 − j (Td/2) Ω) = e^{jω}
∴ ω = ∠(1 + j (Td/2) Ω) − ∠(1 − j (Td/2) Ω)
= tan⁻¹ (TdΩ/2) − (− tan⁻¹ (TdΩ/2))
= 2 tan⁻¹ (TdΩ/2)
⇒ z = (1 + Tds/2) / (1 − Tds/2)
Im[s]
jΩ
Re[s]
Im[z]
ω
Re[z]
If z = f(s) is analytic

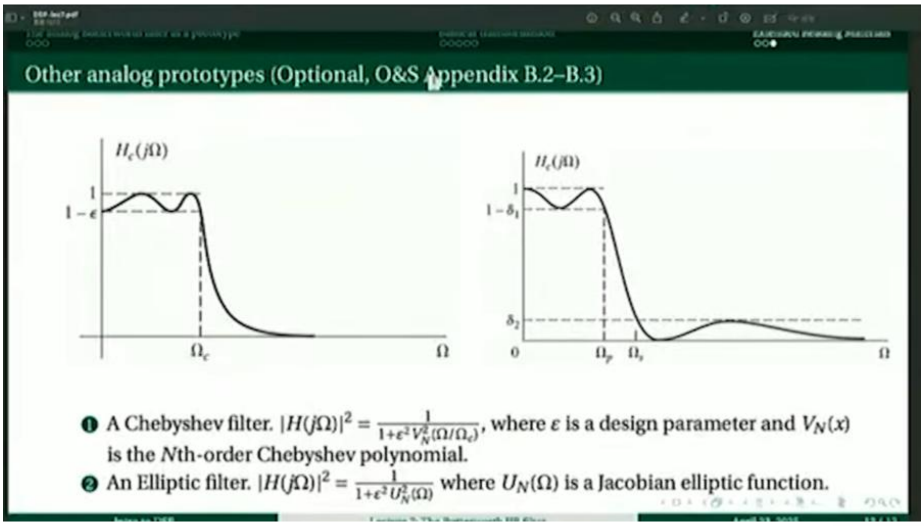

Im[s]
analytic
z = (1 + (Td/2) s) / (1 − (Td/2) s)
Re[s]
Im[z]
Im[s]
Ωc
Re[s]
z(s+ds1)
Re[z]
z(s)
z = e^jω


Implication of Ω(ω) = (2/Td) tan(ω/2)
Figure 1: The frequency response G(e^jω) = H(jΩ(ω)). Note that Ω ∈ [0,∞) is mapped to 0 ≤ ω < π.


If s = jΩ
z = (1 + j (Td/2) Ω) / (1 − j (Td/2) Ω) = e^jω
∴ |z| = 1
xc(t)
A/D
x[n] = xc(nT)
T
ω = ∠(1 + j (Td/2) Ω) − ∠(1 − j (Td/2) Ω)
= tan⁻¹ (TdΩ/2) − (− tan⁻¹ (TdΩ/2))
= 2 tan⁻¹ (TdΩ/2)
Ω = ωT
z = (1 + Td s/2) / (1 − Td s/2)
Im[s]
Re[s]
Im[z]
Re[z]
If f(s) is analytic


Butter-worth
$H(j\Omega)$
Ω_c
$\Omega(\omega)$
$\Omega = \frac{2}{T_d} \tan \frac{\omega}{2}$
$G(e^{j\omega})$
$1/\sqrt{2}$
ω_c
ω
π


History of the Butterworth filter (Source: Wikipedia)
V_{in}
L_1
L_3
V_{out}
C_2
R_4
Figure 2: An implementation of the 3rd-order Butterworth filter. The filter becomes a Butterworth filter with cutoff frequency $\Omega_c = 1$ rad/s when $C_2 = 4/3$ F, $R_4 = 1\Omega$, $L_1 = 3/2$ H and $L_3 = 1/2$ H. Stephen Butterworth (1885-1958) was a British engineer and physicist.


Other analog prototypes (Optional, O&S Appendix B.2–B.3)
$H_c(j\Omega)$
1
$1-\epsilon$
Ω_c
Ω
$H_c(j\Omega)$
1
$1-\delta_1$
δ_2
0
Ω_p
Ω_s
Ω
A Chebyshev filter. $|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 V_N^2(\Omega/\Omega_c)}$, where ϵ is a design parameter and $V_N(x)$ is the Nth-order Chebyshev polynomial.
An Elliptic filter. $|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 U_N^2(\Omega)}$ where $U_N(\Omega)$ is a Jacobian elliptic function.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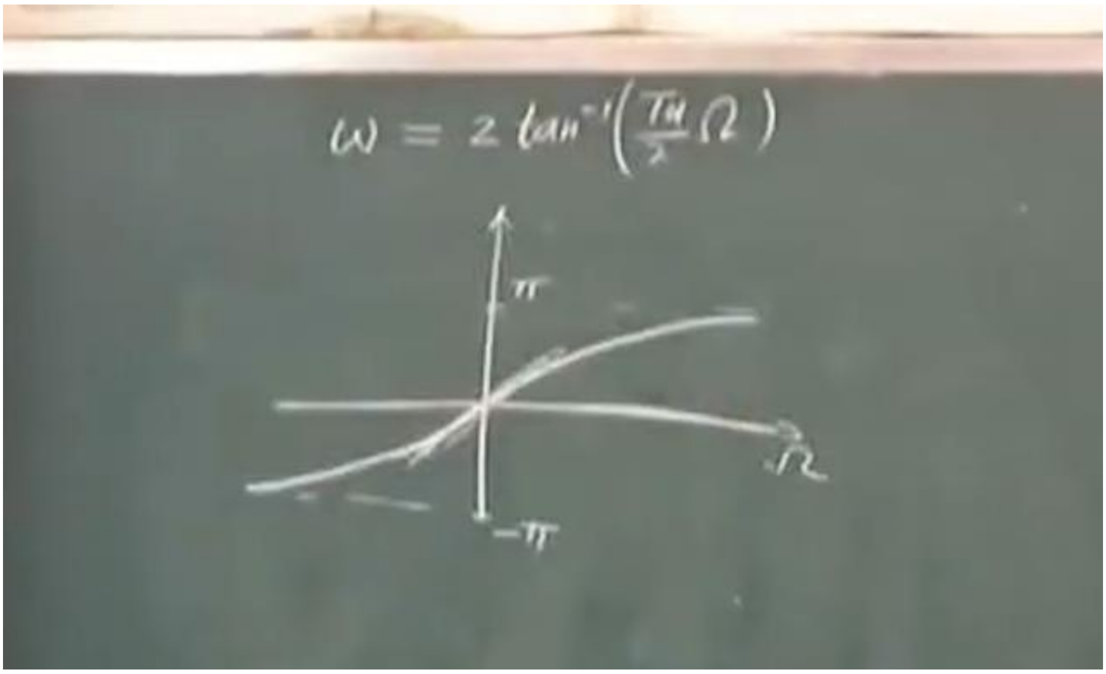
$\omega = 2\tan^{-1}\left(\frac{T_d}{2}\Omega\right)$
π
−π
Ω