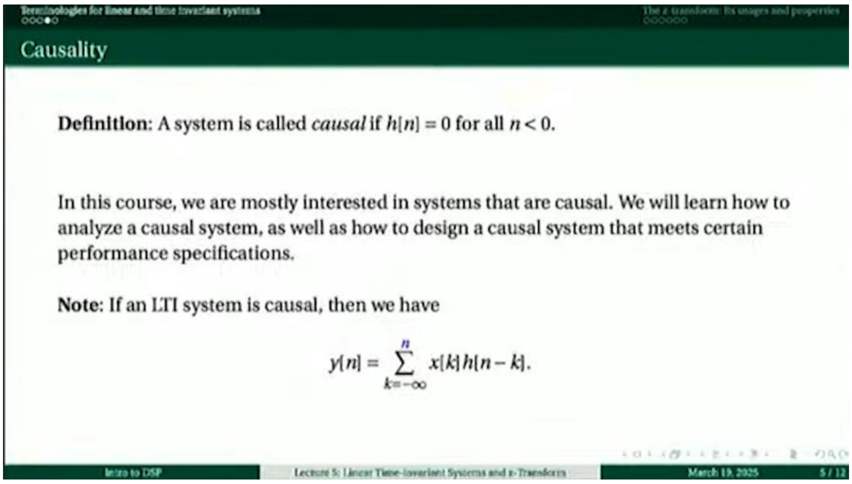
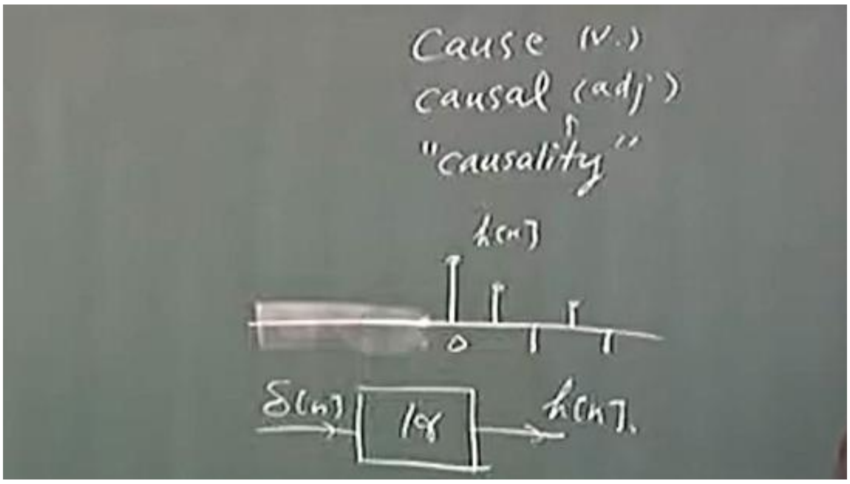
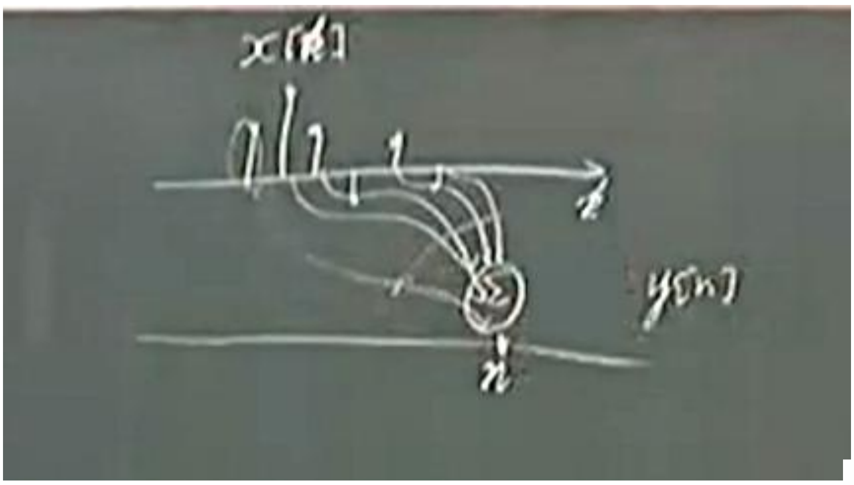
Lecture5 Linear Time-invariant Systems and z-Transform







Examples of a causal LTI system

We can specify an important class of causal LTI systems by *difference equations*. An example would be

$$y[n] = y[n-1] + y[n-2] + x[n]$$

implemented iteratively in time, perhaps using a while loop, assuming that $y[n] = 0$ for $n < 0$.

Exercise: Is this system linear, time-invariant, and causal?

More generally, we will study all *linear constant-coefficient difference equations* (LCCDEs) in the next lecture.

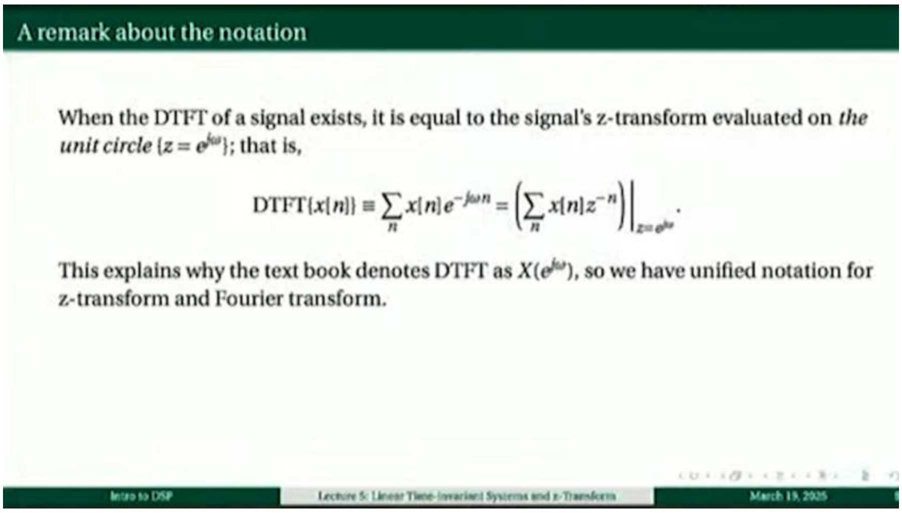
1. Terminologies for linear and time invariant systems
2. The z-transform: Its usages and properties

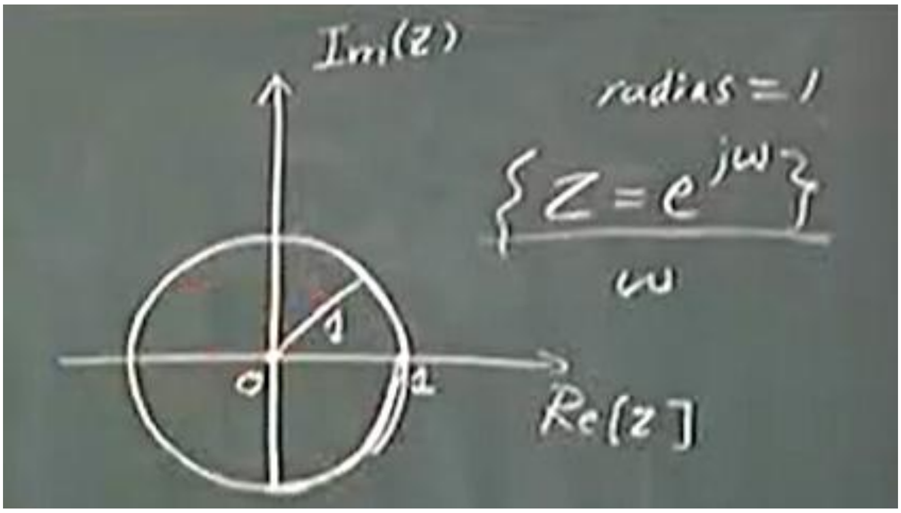
Z-transform

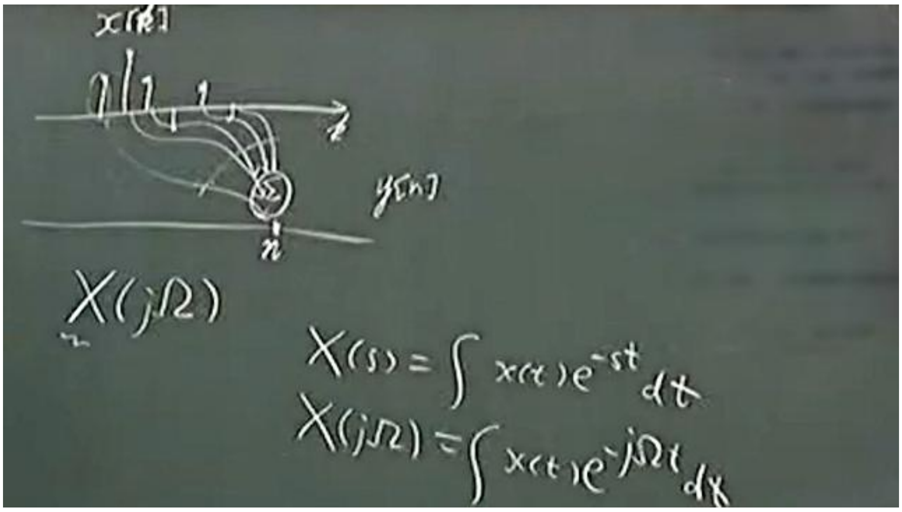
Definition: For any given sequence $\{p[n], n \in \mathbb{Z}\}$, one can attempt to evaluate the following expression

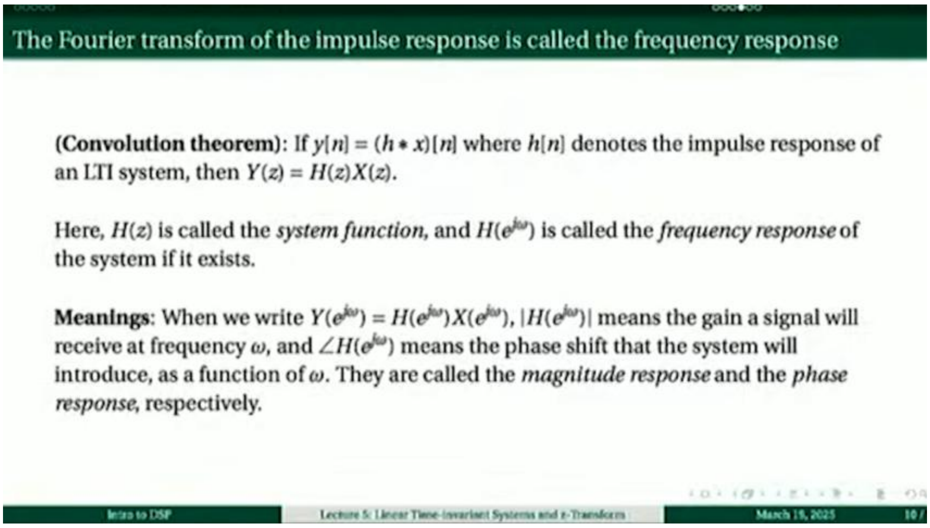
$$P(z) := \sum_{n=-\infty}^{\infty} p[n] z^{-n} \qquad (2)$$

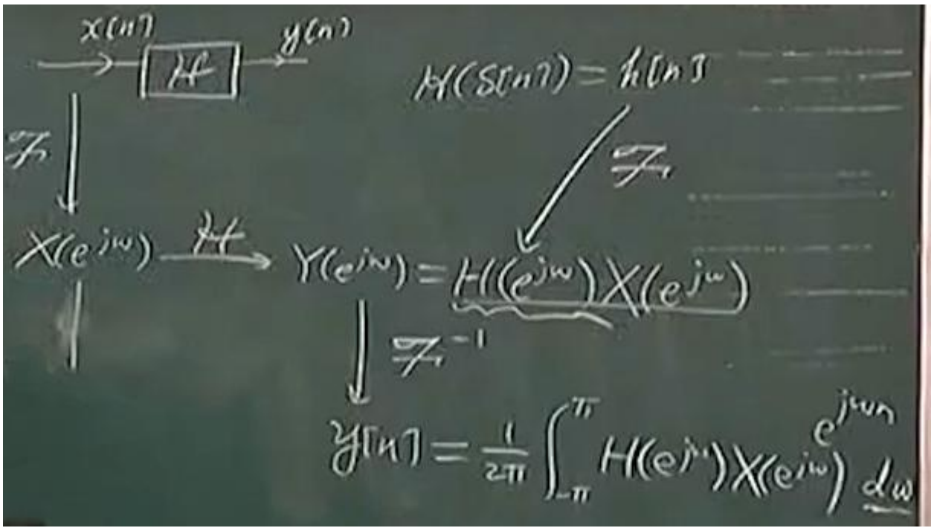
on the complex plane $z \in \mathbb{C}$. $P(z)$ is called the *z-transform* of $p[n]$ which is well-defined over the region where Eq. (2) converges.

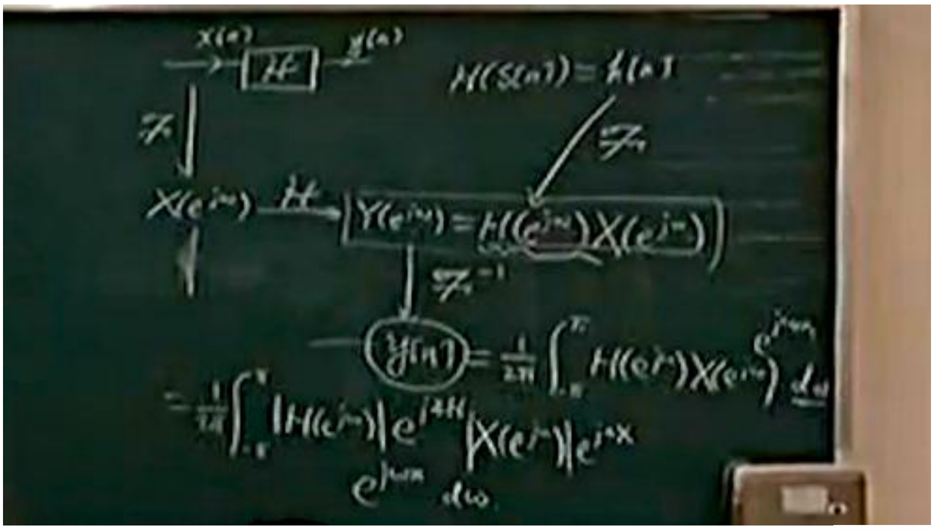

A remark about the notation
When the DTFT of a signal exists, it is equal to the signal's z-transform evaluated on *the unit circle* $\{z = e^{j\omega}\}$; that is,
$$\mathrm{DTFT}\{x[n]\} \equiv \sum_n x[n]e^{-j\omega n} = \left(\sum_n x[n]z^{-n}\right)\Big|_{z=e^{j\omega}}.$$
This explains why the text book denotes DTFT as $X(e^{j\omega})$, so we have unified notation for z-transform and Fourier transform.
Intro to DSP
Lecture 5: Linear Time-Invariant Systems and z-Transform
March 19, 2025

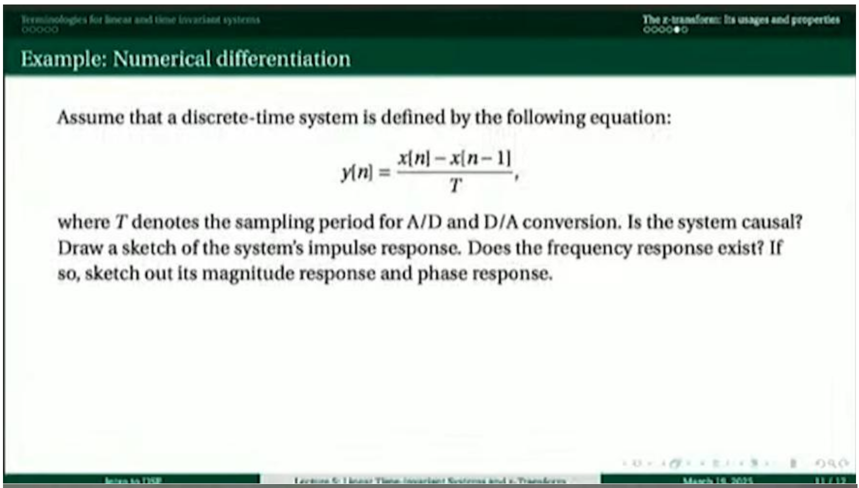

Im(z)
radius = 1
$\{z = e^{j\omega}\}$
ω
1
0
Re(z)

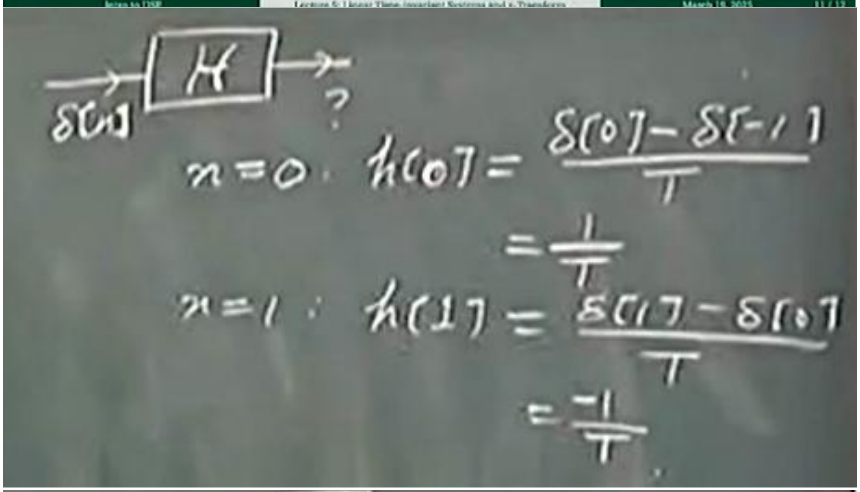

x[k]
k
y[n]
n
$X(j\Omega)$
$X(s) = \int x(t)e^{-st}dt$
$X(j\Omega) = \int x(t)e^{-j\Omega t}dt$

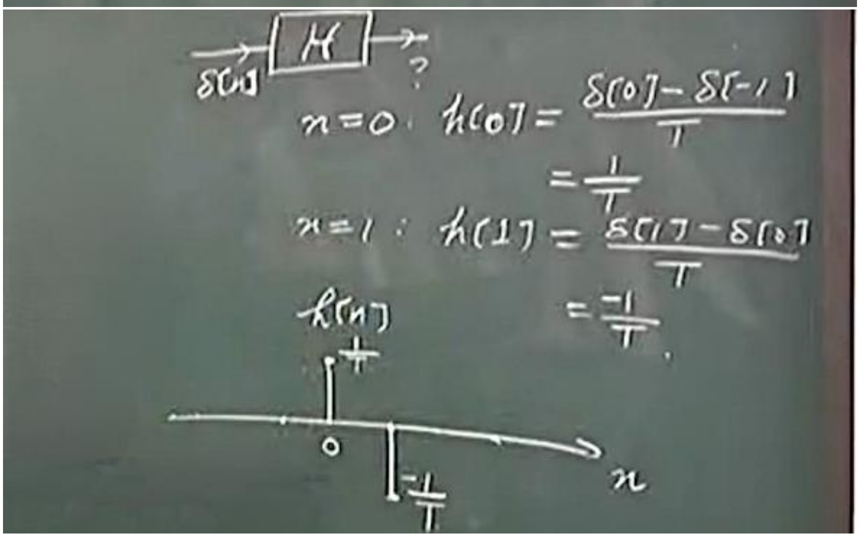

The Fourier transform of the impulse response is called the frequency response
(Convolution theorem): If $y[n] = (h * x)[n]$ where $h[n]$ denotes the impulse response of an LTI system, then $Y(z) = H(z)X(z)$.
Here, $H(z)$ is called the *system function*, and $H(e^{j\omega})$ is called the *frequency response* of the system if it exists.
Meanings: When we write $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$, $|H(e^{j\omega})|$ means the gain a signal will receive at frequency ω, and $\angle H(e^{j\omega})$ means the phase shift that the system will introduce, as a function of ω. They are called the *magnitude response* and the *phase response*, respectively.
Intro to DSP
Lecture 5: Linear Time-invariant Systems and z-Transform
March 19, 2025


$x[n] \rightarrow \boxed{\mathcal{H}} \rightarrow y[n]$
$\mathcal{H}(\delta[n]) = h[n]$
$\mathcal{F}$
$X(e^{j\omega}) \xrightarrow{\mathcal{H}} Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
$\mathcal{F}^{-1}$
$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})X(e^{j\omega})e^{j\omega n}\,d\omega$


$x[n] \rightarrow \boxed{\mathcal{H}} \rightarrow y[n]$
$\mathcal{H}(\delta[n]) = h[n]$
$\mathcal{F}$
$X(e^{j\omega}) \xrightarrow{\mathcal{H}} \boxed{Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})}$
$\mathcal{F}^{-1}$
$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})X(e^{j\omega})e^{j\omega n}\,d\omega$
$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|e^{j\angle H}|X(e^{j\omega})|e^{j\angle X}e^{j\omega n}\,d\omega$


Terminologies for linear and time invariant systems
The z-transform: Its usages and properties
Example: Numerical differentiation
Assume that a discrete-time system is defined by the following equation:
$y[n] = \frac{x[n] - x[n-1]}{T}$,
where T denotes the sampling period for A/D and D/A conversion. Is the system causal? Draw a sketch of the system's impulse response. Does the frequency response exist? If so, sketch out its magnitude response and phase response.


H
δ[n]
?
n=0 : h[0] = (δ[0] − δ[−1]) / T
= 1/T
n=1 : h[1] = (δ[1] − δ[0]) / T
= −1/T

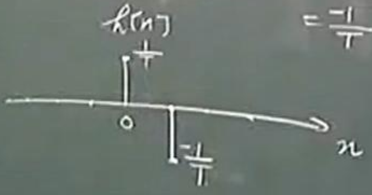

H
δ[n]
?
n=0 : h[0] = (δ[0] − δ[−1]) / T
= 1/T
n=1 : h[1] = (δ[1] − δ[0]) / T
= −1/T
h[n]
1/T
0
−1/T
n

$$H(e^{j\omega}) = \sum_{n} h[n] e^{-j\omega n}$$
$$= h[0]e^{-j\omega 0} + h[1]e^{-j\omega 1}$$
$$= \frac{1}{T} 1 - \frac{1}{T} e^{-j\omega}$$
$$= \frac{1}{T}(1 - e^{-j\omega})$$

$$\delta[n] \rightarrow \boxed{H} \rightarrow ?$$

$n=0$: $h[0] = \frac{\delta[0]-\delta[-1]}{T} = \frac{1}{T}$

$n=1$: $h[1] = \frac{\delta[1]-\delta[0]}{T} = \frac{-1}{T}$



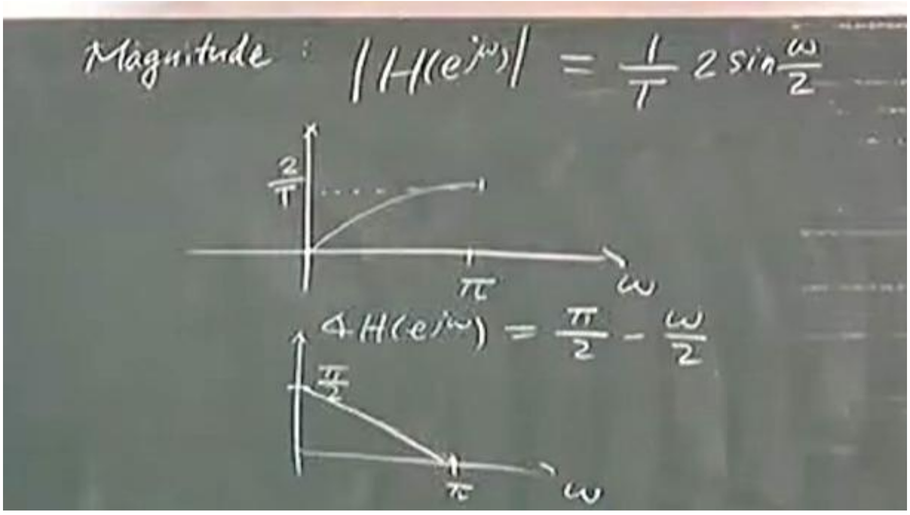
Sol 2

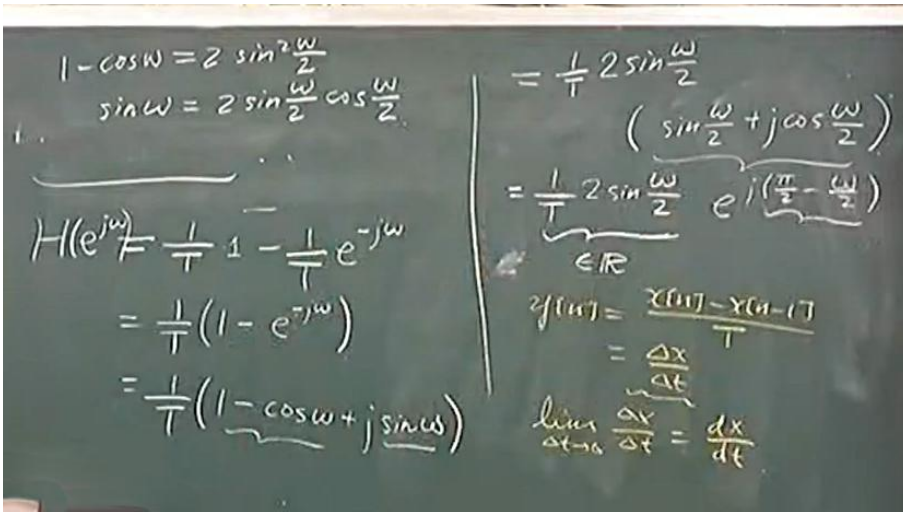
$$y[n] = \frac{x[n] - x[n-1]}{T}$$
$$\Big| Z$$
$$Y(z) = \frac{1}{T}\left(X(z) - z^{-1}X(z)\right)$$
$$= \frac{1}{T}(1 - z^{-1})X(z)$$
$$\equiv H(z)X(z)$$
$$H(z) = \frac{1 - z^{-1}}{T}$$
$$H(e^{j\omega}) = \frac{1 - e^{-j\omega}}{T}$$

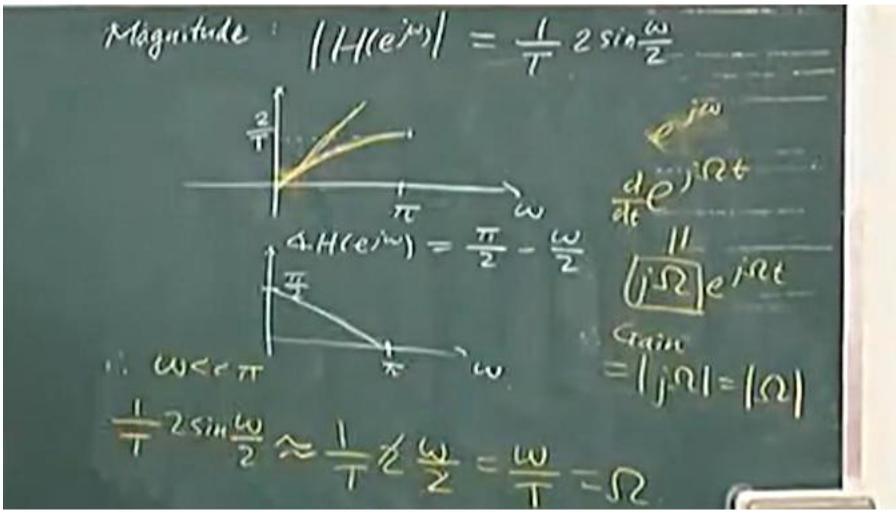
$$1 - \cos\omega = 2\sin^2\frac{\omega}{2}$$
$$\sin\omega = 2\sin\frac{\omega}{2}\cos\frac{\omega}{2}$$

$$H(e^{j\omega}) = \frac{1}{T} 1 - \frac{1}{T} e^{-j\omega}$$
$$= \frac{1}{T}(1 - e^{-j\omega})$$
$$= \frac{1}{T}(1 - \cos\omega + j\sin\omega)$$

$$= \frac{1}{T} 2\sin\frac{\omega}{2}\left(\sin\frac{\omega}{2} + j\cos\frac{\omega}{2}\right)$$
$$= \underbrace{\frac{1}{T} 2\sin\frac{\omega}{2}}_{\in \mathbb{R}} e^{j\left(\frac{\pi}{2} - \frac{\omega}{2}\right)}$$


Magnitude : $|H(e^{j\omega})| = \frac{1}{T} 2\sin\frac{\omega}{2}$
$\frac{2}{T}$
π
ω
$\measuredangle H(e^{j\omega}) = \frac{\pi}{2} - \frac{\omega}{2}$
$\frac{\pi}{2}$
π
ω


$1-\cos\omega = 2\sin^2\frac{\omega}{2}$
$\sin\omega = 2\sin\frac{\omega}{2}\cos\frac{\omega}{2}$
$H(e^{j\omega}) = \frac{1}{T}1 - \frac{1}{T}e^{-j\omega}$
$= \frac{1}{T}(1-e^{-j\omega})$
$= \frac{1}{T}(1-\cos\omega + j\sin\omega)$
$= \frac{1}{T}2\sin\frac{\omega}{2}\left(\sin\frac{\omega}{2} + j\cos\frac{\omega}{2}\right)$
$= \underbrace{\frac{1}{T}2\sin\frac{\omega}{2}}_{\in\mathbb{R}} e^{j(\frac{\pi}{2}-\frac{\omega}{2})}$
$y[n] = \frac{x[n]-x[n-1]}{T}$
$= \frac{\Delta x}{\Delta t}$
$\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} = \frac{dx}{dt}$


Magnitude : $|H(e^{j\omega})| = \frac{1}{T} 2\sin\frac{\omega}{2}$
$\frac{2}{T}$
π
ω
$\measuredangle H(e^{j\omega}) = \frac{\pi}{2} - \frac{\omega}{2}$
$\frac{\pi}{2}$
π
ω
$e^{j\omega}$
$\frac{d}{dt}e^{j\Omega t}$
$\boxed{j\Omega}e^{j\Omega t}$
Gain
$= |j\Omega| = |\Omega|$
$\therefore\ \omega \ll \pi$
$\frac{1}{T}2\sin\frac{\omega}{2} \approx \frac{1}{T}\cancel{2}\frac{\omega}{\cancel{2}} = \frac{\omega}{T} = \Omega$

Thank you for listening

1. The next lecture will be about *difference equations* 差分方程, and solving them using z-transform.

2. You can find Prof. Liu on Instagram and Threads (`ID='lineartimeinvariant'`)