Lecture3 Selected Properties of Discrete-time Fourier Transform

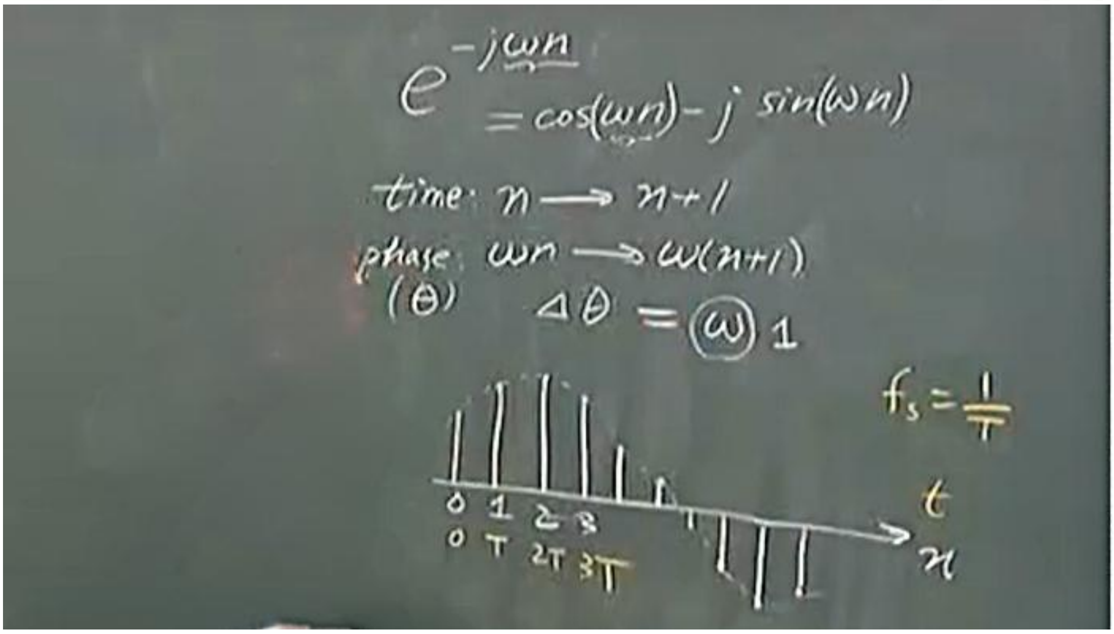
Motivation and Definition

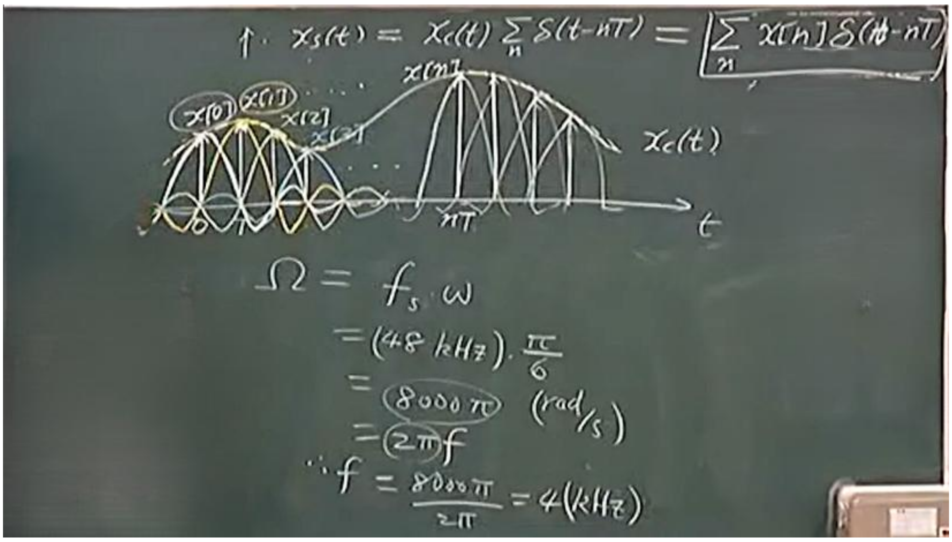
Definition: For any sequence $x[n]$, we define its *discrete-time Fourier transform* (DTFT) as follows,

$$X(e^{j\omega}) := \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

Remarks: The unit of ω can be regarded as rad/sample. The signal $x[n]$ should be seen as the outcome of sampling a continuous-time signal $x(t)$; that is, $x[n] = x(nT)$ with a non-specified sampling period T. To convert between ω (rad/sample) and Ω (rad/s), use the relation $\omega = T\Omega$.

Exercise: If sampling rate is 48 kHz, which frequency in Hz does $\omega = \pi/6$ represent?





Inverse Transform

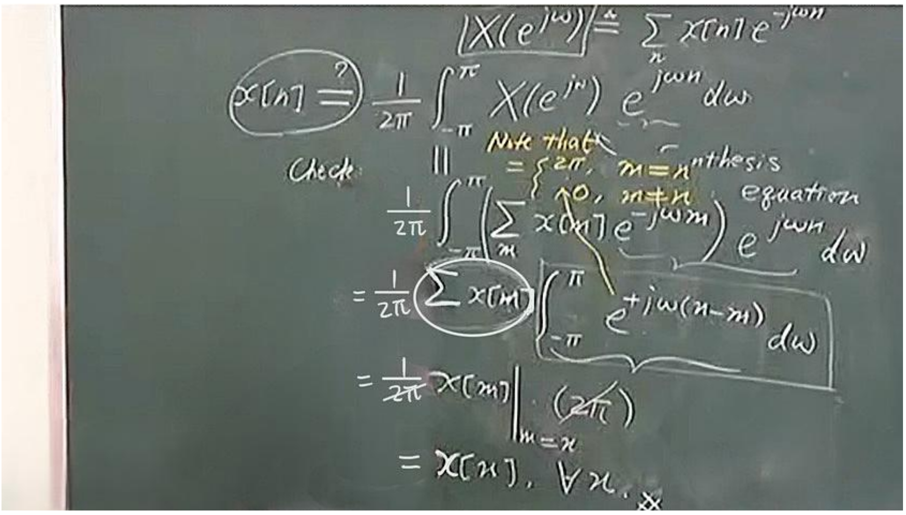
We can show that

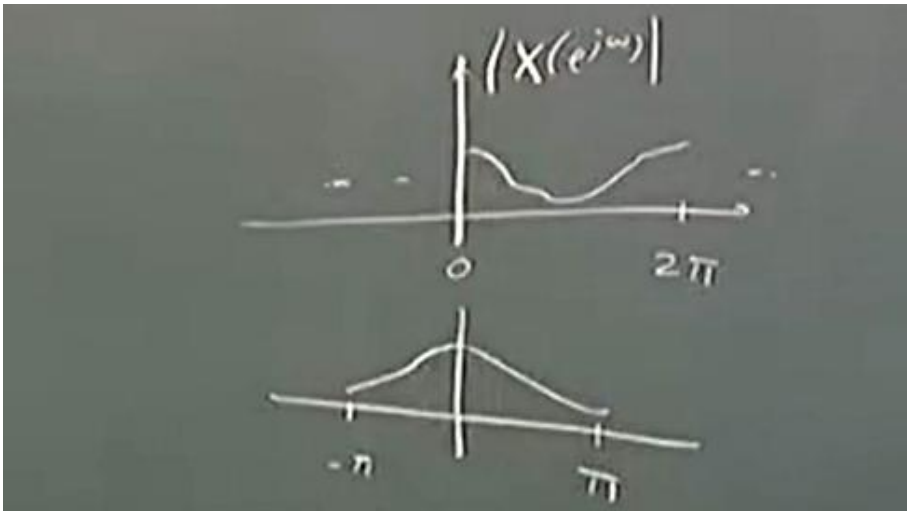
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

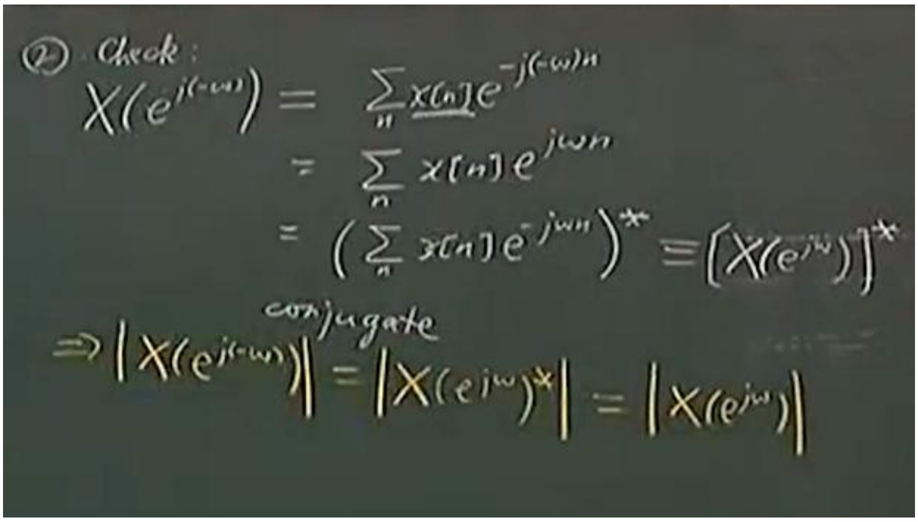
Check:

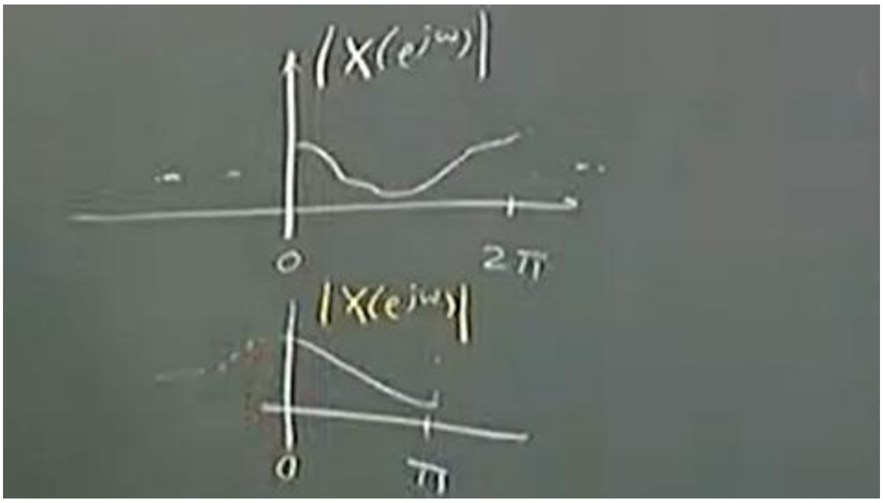
Several Properties (from O&S Sec. 2.8-2.9)

1. Periodicity: $X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$.
2. If $x[n] \in \mathbb{R}, \forall n$, then $X(e^{-j\omega}) = [X(e^{j\omega})]^*$.[1]
3. If $x[n] \in \mathbb{R}$ and $x[-n] = x[n], \forall n$, then $X(e^{j\omega})$ is real.
4. If $y[n] = x[n-m]$ (where m is an integer of course), then $Y(e^{j\omega}) = e^{-j\omega m}X(e^{j\omega})$.
5. **Convolution**: Define
$$y[n] = (x * h)[n] := \sum_k x[k]h[n-k]$$
$y[n]$ is called the *convolution* of $x[n]$ and $h[n]$. Note that $x * h = h * x$ for arbitrary sequences $x[n]$ and $h[n]$. We have
$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$
for all $\omega \in [-\pi, \pi]$. This property is called the *convolution theorem*.


$\boxed{X(e^{j\omega})} \triangleq \sum_n x[n] e^{-j\omega n}$
$x[n] \stackrel{?}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$
synthesis equation
Check:
Note that $= \begin{cases} 2\pi, & m = n \\ 0, & m \neq n \end{cases}$
$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_m x[m] e^{-j\omega m}\right) e^{j\omega n}\, d\omega$
$= \frac{1}{2\pi}\sum_m x[m] \int_{-\pi}^{\pi} e^{+j\omega(n-m)}\, d\omega$
$= \frac{1}{2\pi} x[m]\Big|_{m=n} (2\pi)$
$= x[n], \ \forall n$. ※


$|X(e^{j\omega})|$
0
2π
$-\pi$
π


② Check:
$X(e^{j(-\omega)}) = \sum_n x[n] e^{-j(-\omega)n}$
$= \sum_n x[n] e^{j\omega n}$
$= \left(\sum_n x[n] e^{-j\omega n}\right)^* = [X(e^{j\omega})]^*$
conjugate
$\Rightarrow |X(e^{j(-\omega)})| = |X(e^{j\omega})^*| = |X(e^{j\omega})|$



$$Y(e^{j\omega}) = e^{-j\omega m} X(e^{j\omega})$$

$$\angle Y(e^{j\omega}) = \angle e^{-j\omega m} + \angle X(e^{j\omega})$$

Group delay:

$$\tau_g^{(Y)} = \frac{-\partial \angle Y(e^{j\omega})}{\partial \omega} = -\left\{ \frac{\partial \overbrace{\angle e^{-j\omega m}}^{=-\omega m}}{\partial \omega} + \frac{\partial \angle X(e^{j\omega})}{\partial \omega} \right\}$$

$$= -\left\{ -m + \frac{\partial X(e^{j\omega})}{\partial \omega} \right\}$$

$$= m + \boxed{\tau_g^{(X)}}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x * h$$

$h[n]$: impulse response

$$\boxed{Y(e^{j\omega})} = \sum_n y[n] e^{-j\omega n}$$

$$= \sum_n \left(\sum_k x[k] h[n-k] \right) e^{-j\omega n}$$

$$= \sum_k x[k] \underbrace{\sum_n h[n-k] e^{-j\omega n}}_{= e^{-j\omega k} H(e^{j\omega}) \ (\because \text{Shift})}$$

$$= \underbrace{\left(\sum_k x[k] e^{-j\omega k} \right)}_{= \boxed{X(e^{j\omega})}} \boxed{H(e^{j\omega})}$$

A theorem related to the energy

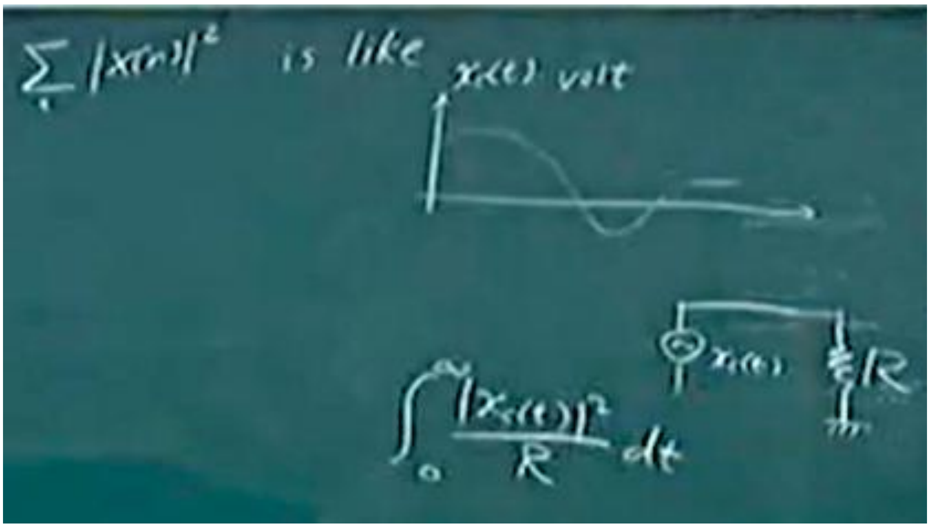
Here is a nice property regarding the *energy* of a signal.

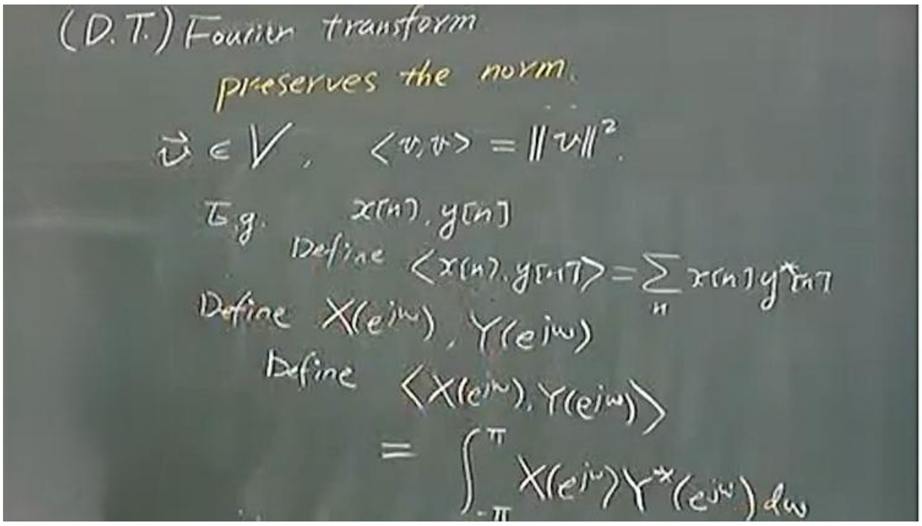
Parseval's theorem

If $\sum_n |x[n]|^2 < \infty$, then

$$\sum_n |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

Implication: Energy can be calculated in the time domain or in the frequency domain.
Contemplation: In the language of linear algebra, we say that Fourier transform....





Extended reading on the existence of DTFT (optional)

DTFT has different notions of "existing".
The first notion is if $\sum_n |x[n]| < \infty$, then $\sum_n x[n]e^{-j\omega n}$ converges for any $\omega \in \mathbb{R}$. This condition is sufficient but not necessary for DTFT to converge.

The second notion is, if $\sum_n |x[n]| = \infty$ but $\sum_n |x[n]|^2 < \infty$, then

$$X(e^{j\omega}) := \lim_{N\to\infty} \sum_{n=-N}^{N} x[n]e^{-j\omega n}$$

converges for all $\omega \in \mathbb{R}$.