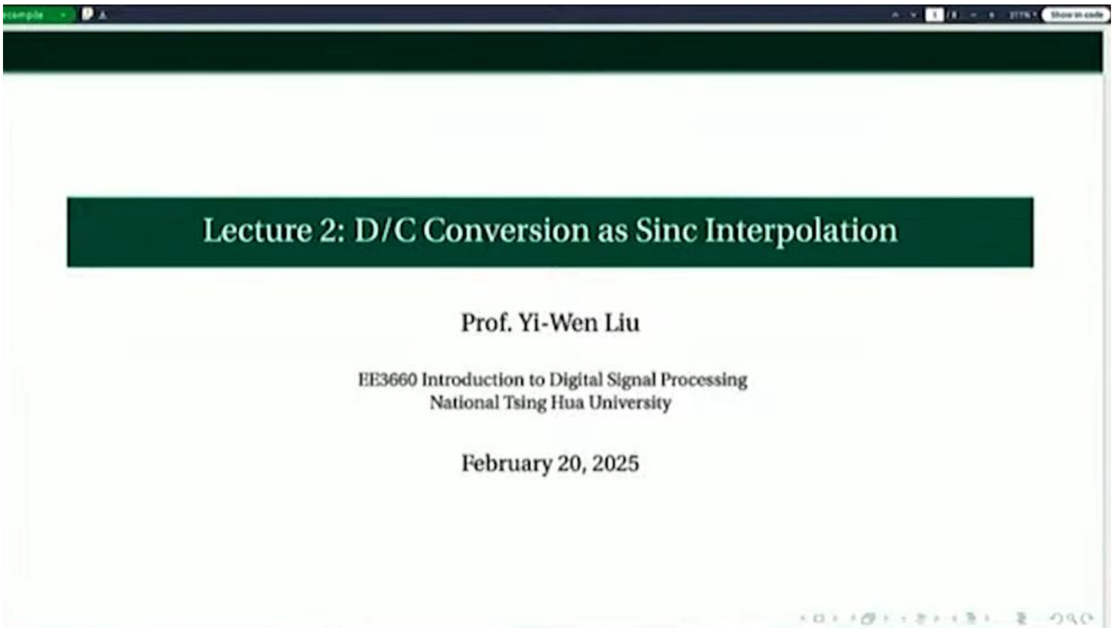
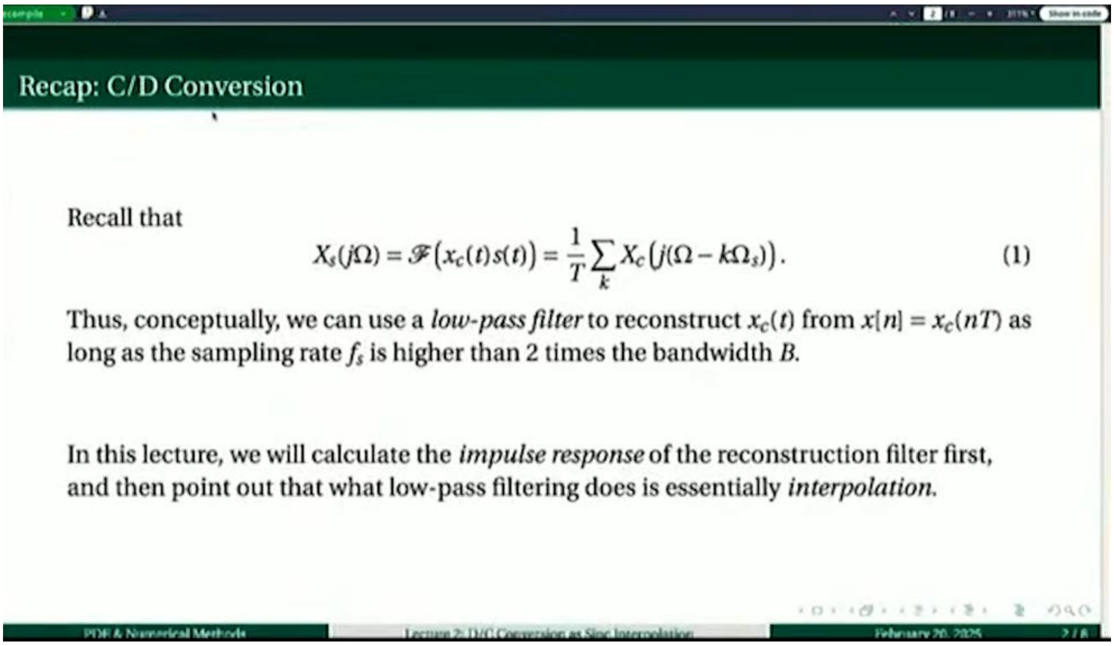
Lecture2 D/Continuous Conversion as Sinc Interpolation





Let's consider an analog filter with a frequency response $H(j\Omega)$ as follows,

$$H(j\Omega) = \begin{cases} T, & |\Omega| < \pi f_s, \\ 0, & \text{otherwise.} \end{cases}$$

Remarks:

1. From Eq. (1), it should be clear that $H(j\Omega)X_s(j\Omega) = X_c(j\Omega)$.
2. This is the action of filtering, which is a multiplication in the frequency domain.
3. By *convolution theorem*, we can achieve filtering by convolving $x_s(t)$ with an *impulse response* $h(t)$, which is the inverse Fourier transform of $H(j\Omega)$ in the time domain.

Derivation of the Inverse Fourier transform of $H(j\Omega)$

In-class exercise: show that

$$h(t) = \mathscr{F}^{-1}\left(H(j\Omega)\right) := \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\Omega)e^{j\Omega t}\,d\Omega = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}},$$

and draw a sketch of $h(t)$.

D/C conversion as Sinc Interpolation

The function $\frac{\sin\theta}{\theta}$ is often called and denoted as the sinc function; that is, we write $\text{sinc}(\theta) = \frac{\sin\theta}{\theta}$. Be careful also to define that $\text{sinc}(0) := \lim_{t\to 0}\frac{\sin(t)}{t} = 1$.

In the previous slide, we showed that $h(t) = \text{sinc}(\pi t/T)$. Can you draw a sketch of convolving $x_s(t) = x_c(t)\sum_n \delta(t - nT)$ with $h(t)$?

Highlight: Write $x_s(t) = \sum_n x[n]\delta(t - nT)$, and thus we have $y(t) = x_s(t) * h(t) = \sum_n x[n]\text{sinc}(t - nT)$, where $*$ denotes convolution in time.

$\boxed{\text{If } x(t) \in \mathbb{R}, \forall t}$

$X(-j\Omega) = X^*(j\Omega)$

$\Rightarrow \begin{cases} |X(-j\Omega)| = |X(j\Omega)| \\ \measuredangle X(-j\Omega) = -\measuredangle X(j\Omega) \end{cases}$

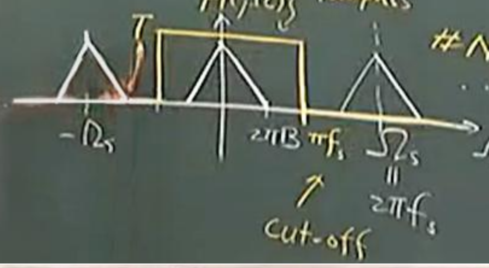
Interpolation 內插

$x_s(t) \triangleq x_c(t)s(t)$,

where

$s(t) = \sum_n \delta(t-nT)$

$X_s(j\Omega) = \frac{1}{T}\sum_k X_c(j(\Omega - k\Omega_s))$



$\mathcal{F}^{-1}(H(j\Omega)) \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\Omega)\, e^{j\Omega t}\, d\Omega$

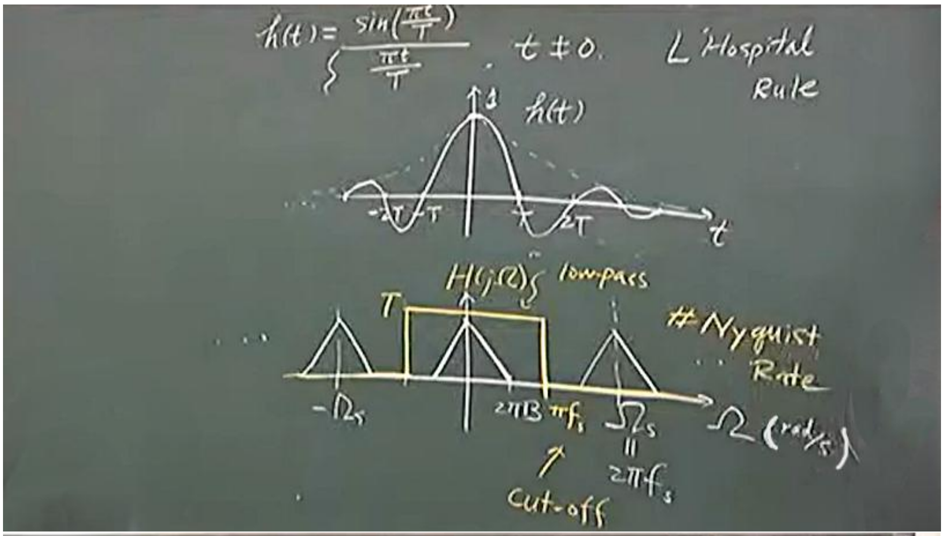
$= \frac{1}{2\pi}\int_{-\pi f_s}^{\pi f_s} T\, e^{j\Omega t}\, d\Omega$

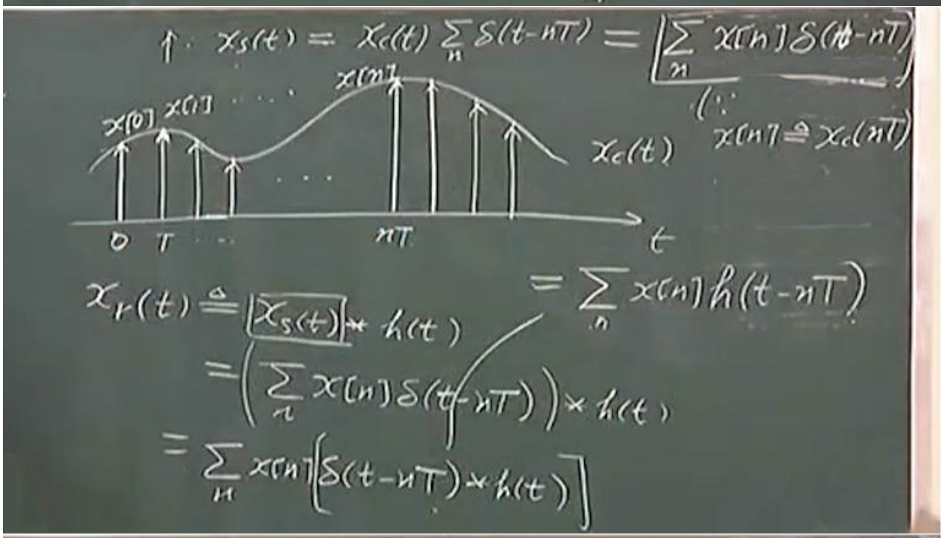
$= \frac{T}{2\pi}\, \frac{1}{jt}\, e^{j\Omega t}\Big|_{-\pi f_s}^{\pi f_s} = \frac{T}{2\pi}\, \frac{1}{jt}\left(e^{j\pi f_s t} - e^{-j\pi f_s t}\right)$

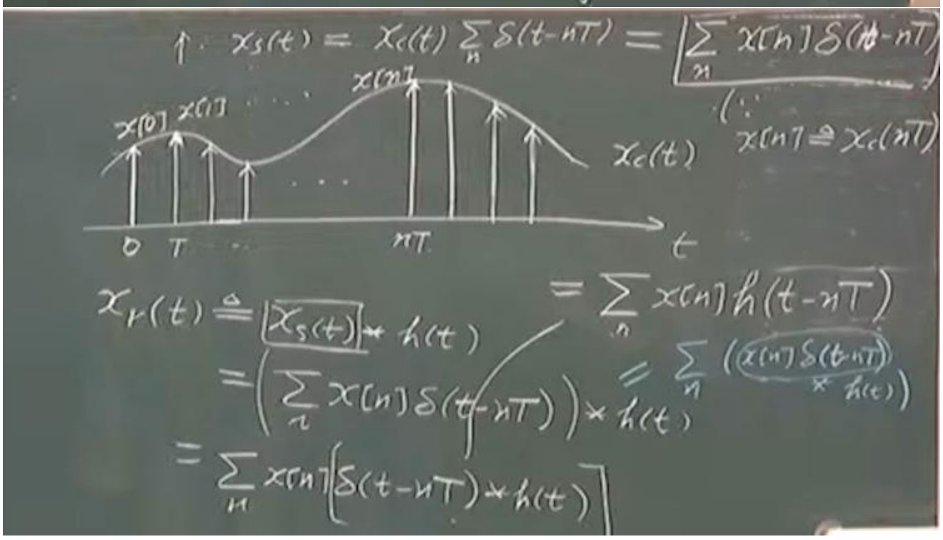
$2j\sin(\pi f_s t)$

$= \frac{T}{\cancel{2}\pi}\, \frac{\cancel{2}\cancel{j}\sin(\pi f_s t)}{\cancel{j}t}$

$= \frac{\sin(\pi/T)t}{\frac{\pi}{T}t}$


$h(t) = \dfrac{\sin(\frac{\pi t}{T})}{\frac{\pi t}{T}}$, $t \neq 0$.
L'Hospital Rule
1
$h(t)$
$-2T$ $-T$ T $2T$ t
$H(j\Omega)$ lowpass
T
$-\Omega_s$ $2\pi B$ πf_s Ω_s $= 2\pi f_s$
Ω (rad/s)
cut-off
Nyquist Rate


$x_s(t) = x_c(t) \sum_n \delta(t-nT) = \sum_n x[n]\delta(t-nT)$
$(\because x[n] \triangleq x_c(nT))$
$x[0]$ $x[1]$ $x[n]$
$x_c(t)$
0 T nT t
$x_r(t) \triangleq \boxed{x_s(t)} * h(t)$
$= \left(\sum_n x[n]\delta(t-nT)\right) * h(t)$
$= \sum_n x[n]\left[\delta(t-nT) * h(t)\right]$
$= \sum_n x[n] h(t-nT)$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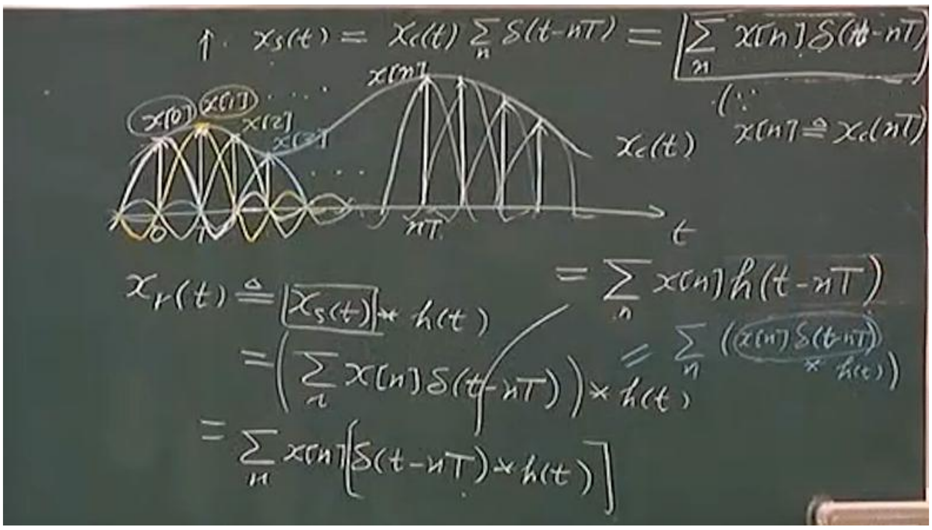
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($\because x[n] \triangleq x_c(nT)$)
x[0] x[1] x[2] x[3] x[n]
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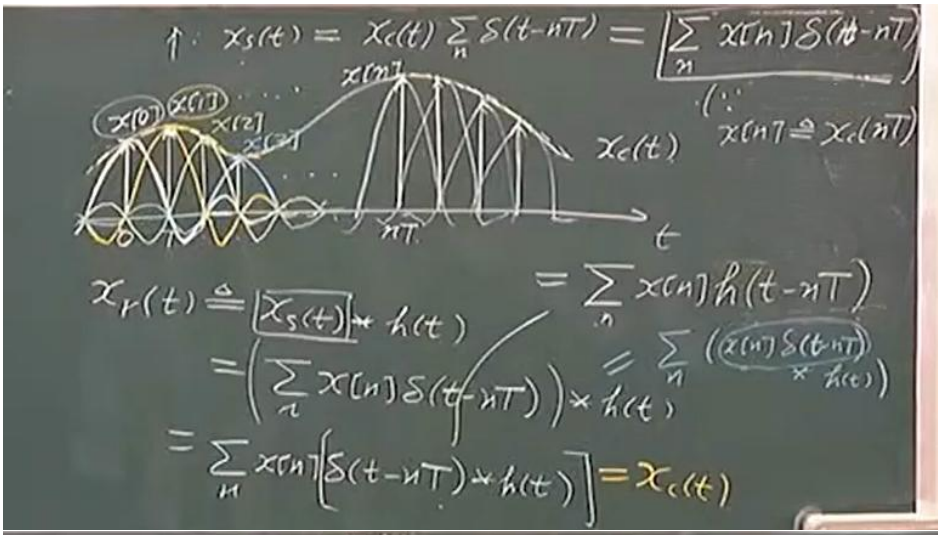
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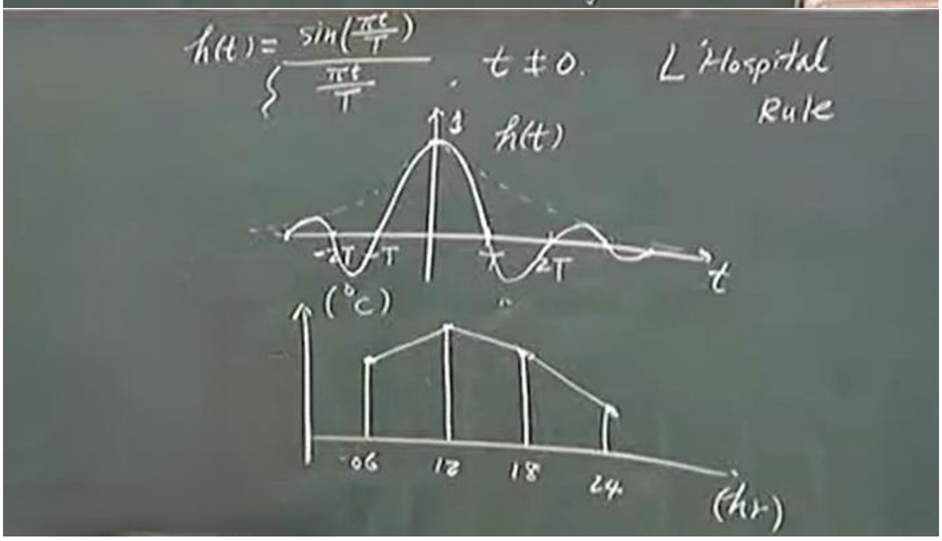
$= \sum_n \left(x[n]\delta(t-nT) * h(t)\right)$
$= \sum_n x[n]h(t-nT)$
$= \sum_n x[n]\left[\delta(t-nT) * h(t)\right] = x_c(t)$


$h(t) = \frac{\sin(\frac{\pi t}{T})}{\frac{\pi t}{T}}$, $t \neq 0$.
L'Hospital Rule
1
h(t)
-2T -T T 2T t
(°C)
06 12 18 24
(hr)

Additional Remarks from the Textbook (1/3)

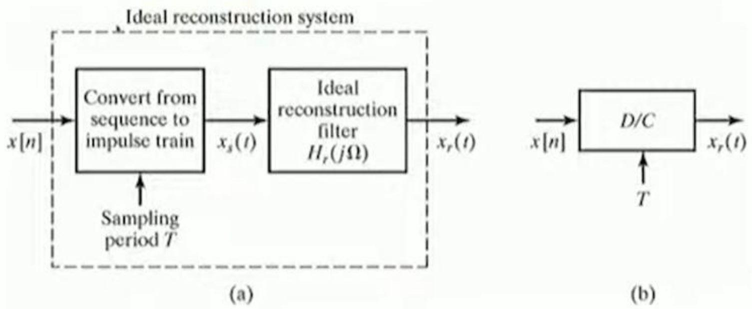


Figure 1: The sinc filter is referred to as the *ideal reconstruction filter.*

Additional Remarks from the Textbook (2/3)

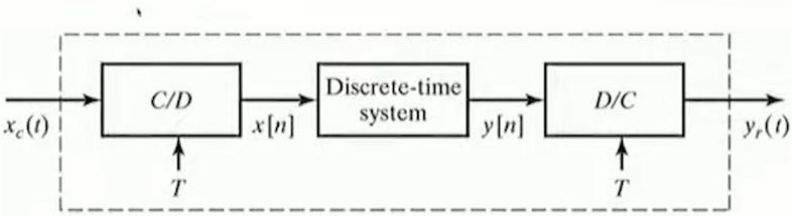


Figure 2: The idea of processing continuous-time signal in the discrete-time domain.
Discussion: What are the advantages?

Additional Remarks from the Textbook (3/3)

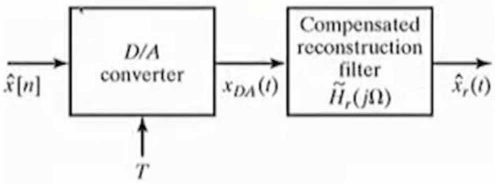


Figure 3: (Optional) The actual practice of D/C conversion. D = Digital, A = Analog. Refer t
Sec. 4.8.4 of O&S if interested.