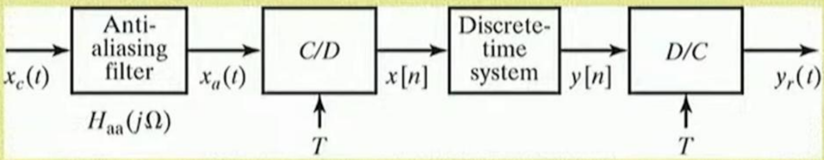
Lecture1 Discrete-time Sampling of Continuous Signals

5

Overall workflow of DSP (C = continuous, D = discrete)



Filters with arbitrary *frequency response* can be implemented by the following equation:

$$y[n] = -\sum_{k=1}^{M} a_k y[n-k] + \sum_{k=0}^{N} b_k x[n-k]$$

6

Different notions of Fourier transform (FT)

- The continuous-time FT $$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$
- The discrete-time FT $$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$
  - what for? #impulseresponse
  - #lineartimeinvariant #convolution
  - The Discrete FT
    - The Fast FT (FFT)
    - The short-time FT (STFT)

Lecture 1: Discrete-time Sampling of Continuous Signals

Prof. Yi-Wen Liu

EE3660 Introduction to Digital Signal Processing
National Tsing Hua University

February 19, 2025

Sampling in the time domain — Effects in the frequency domain

1. Sampling in the time domain
2. Effects in the frequency domain

Definition and Notations

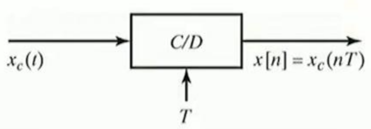
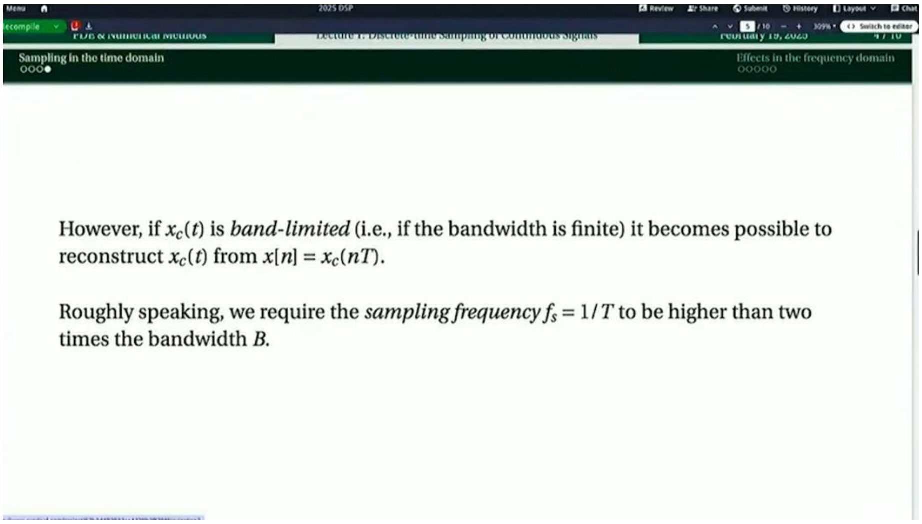
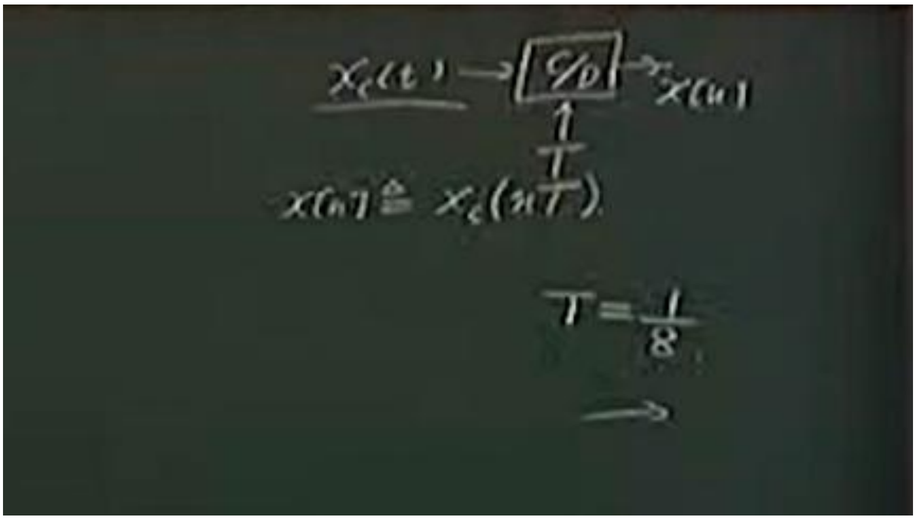


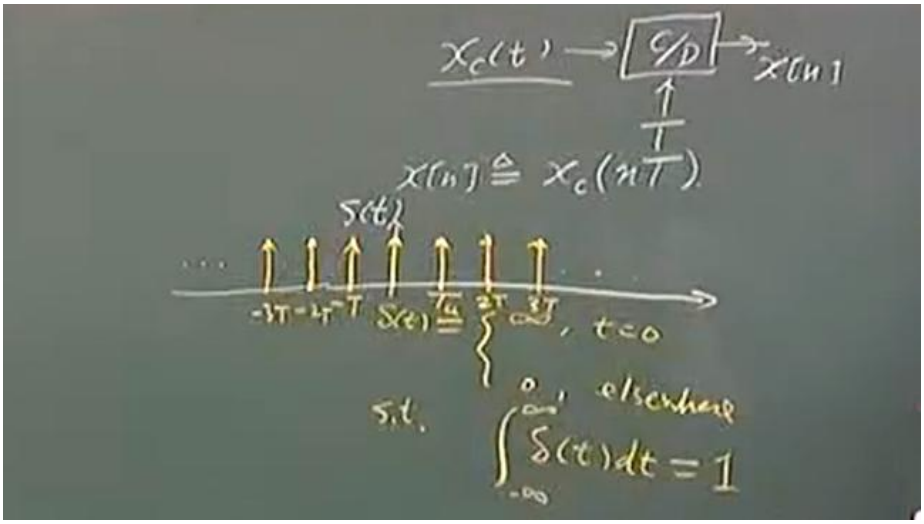
Figure 1: A system diagram of continuous to discrete time conversion. T: *sampling period*; $n \in \mathbb{Z}$.

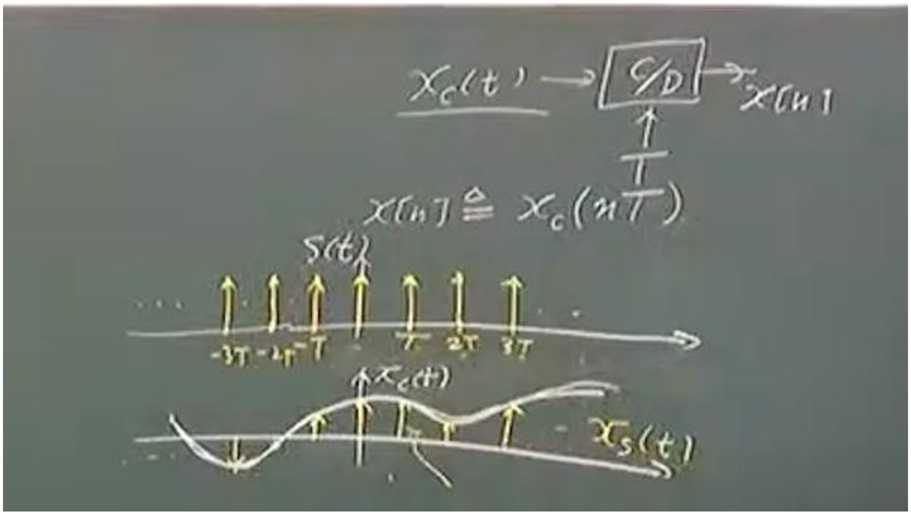
Remarks: $x[n]$ is not yet called *digital*, because arbitrary real numbers cannot be stored by a computer. We will come back to this when we discuss *quantization*.

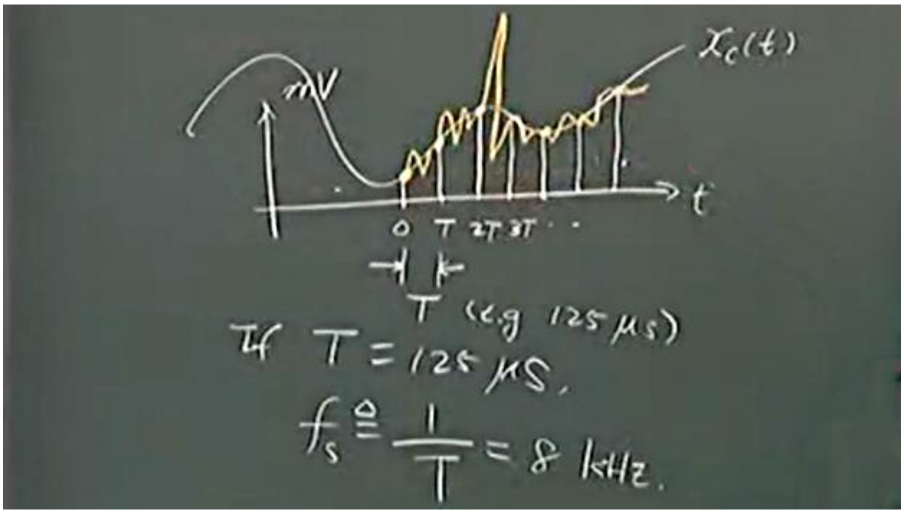
Question: Is C/D invertible?

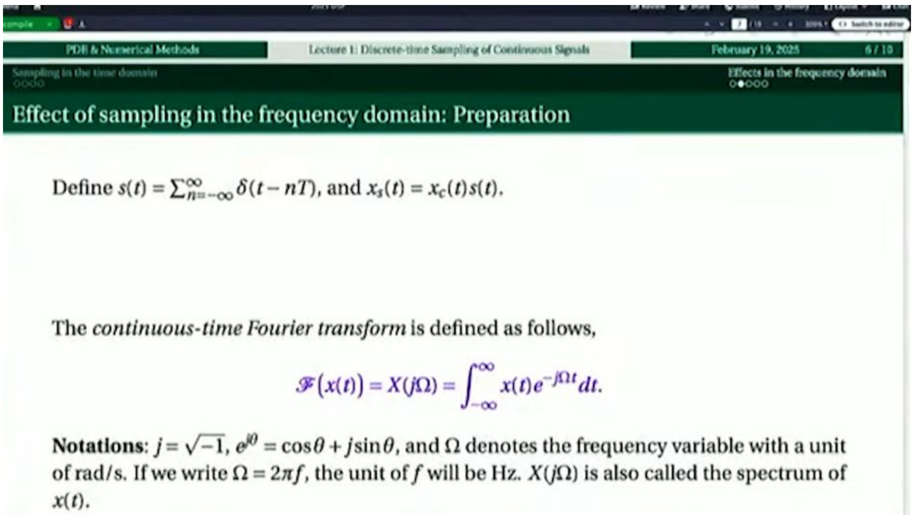

Sampling in the time domain
Effects in the frequency domain
However, if $x_c(t)$ is *band-limited* (i.e., if the bandwidth is finite) it becomes possible to reconstruct $x_c(t)$ from $x[n] = x_c(nT)$.
Roughly speaking, we require the *sampling frequency* $f_s = 1/T$ to be higher than two times the bandwidth B.


$x_c(t) \rightarrow$ C/D $\rightarrow x[n]$
T
$x[n] \triangleq x_c(nT)$
$T = \frac{1}{8}$


$x_c(t) \rightarrow$ C/D $\rightarrow x[n]$
T
$x[n] \triangleq x_c(nT)$
$s(t)$
-3T -2T -T 0 T 2T 3T
$\delta(t) \triangleq \{ \infty, t=0; 0, \text{elsewhere}$
s.t. $\int_{-\infty}^{\infty} \delta(t)dt = 1$


$x_c(t)$
C/D
$x[n]$
T
$x[n] \triangleq x_c(nT)$
$s(t)$
-3T -2T -T
T 2T 3T
$x_c(t)$
$x_s(t)$


$x_c(t)$
t
0 T 2T 3T
T (e.g. 125 μs)
If T = 125 μs,
$f_s \triangleq \frac{1}{T} = 8$ kHz.


PDE & Numerical Methods
Lecture 1: Discrete-time Sampling of Continuous Signals
February 19, 2025
6 / 10
Sampling in the time domain
Effects in the frequency domain
Effect of sampling in the frequency domain: Preparation
Define $s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$, and $x_s(t) = x_c(t)s(t)$.
The *continuous-time Fourier transform* is defined as follows,
$$\mathscr{F}(x(t)) = X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt.$$
Notations: $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j\sin\theta$, and Ω denotes the frequency variable with a unit of rad/s. If we write $\Omega = 2\pi f$, the unit of f will be Hz. $X(j\Omega)$ is also called the spectrum of $x(t)$.

Sampling in the time domain — Effects in the frequency domain

Derivation of aliasing in the frequency domain

Recall $s(t) := \sum_n \delta(t - nT)$.[1] It turns out that its Fourier transform is

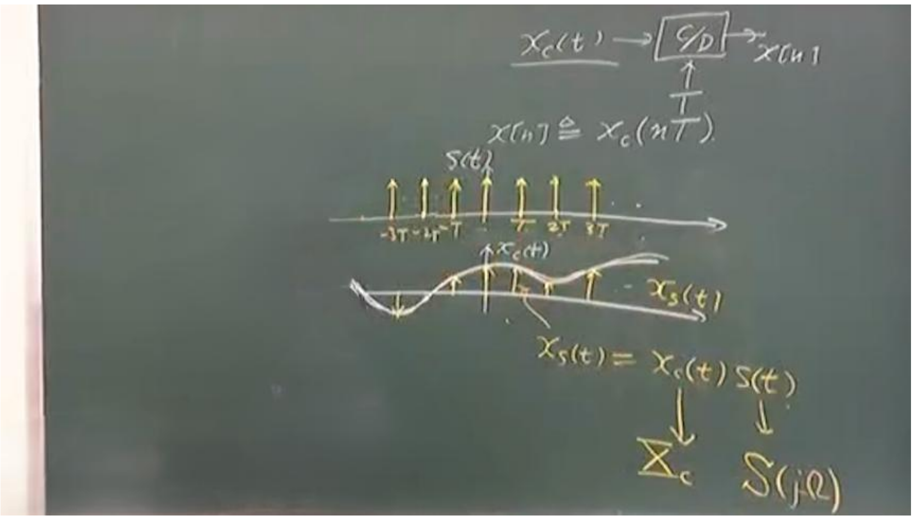
$$S(j\Omega) := \mathscr{F}\big(s(t)\big) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right).$$

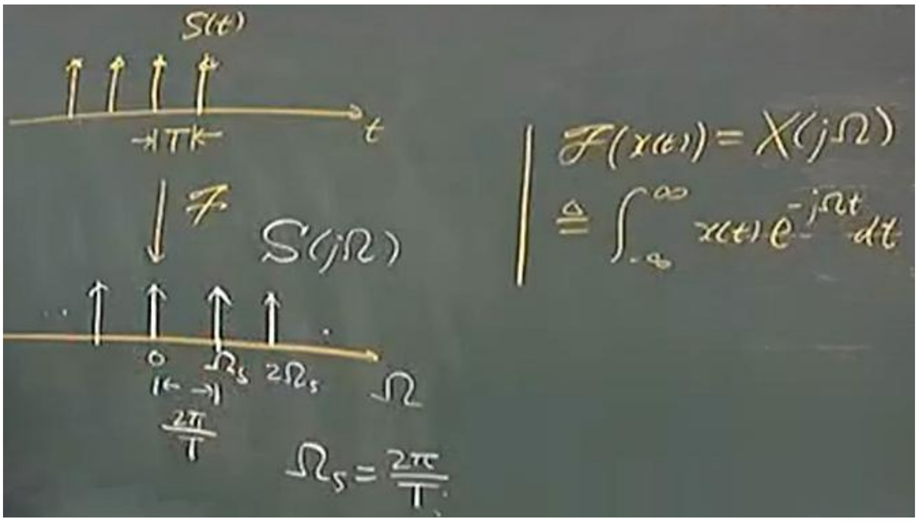
i.e., we can write $s(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \exp\left(jk\frac{2\pi}{T}t\right)$. Therefore,

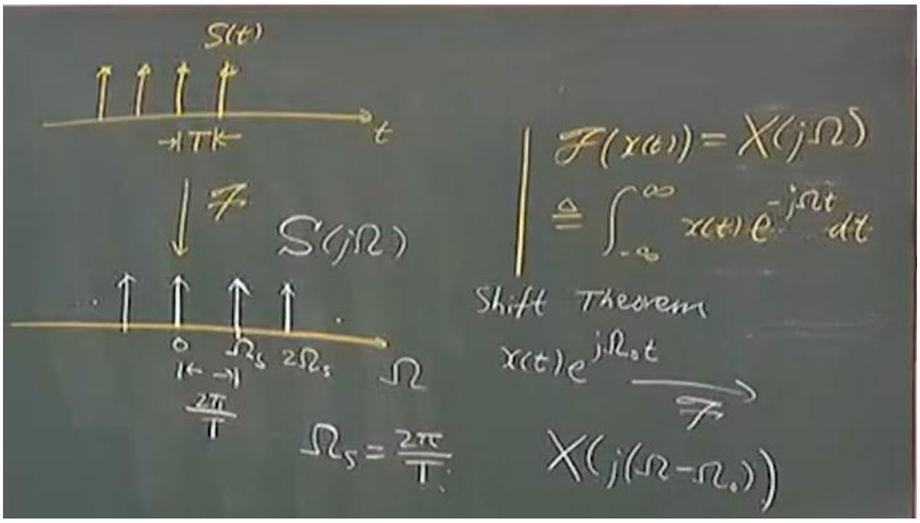
$$\begin{aligned} X_s(j\Omega) &= \mathscr{F}\big(x_c(t)s(t)\big) && (1)\\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} \mathscr{F}\left(x_c(t)\exp\left(jk\frac{2\pi}{T}t\right)\right) && (2)\\ &= \frac{1}{T}\sum_k X_c\big(j(\Omega - k\Omega_s)\big). && (3) \end{aligned}$$

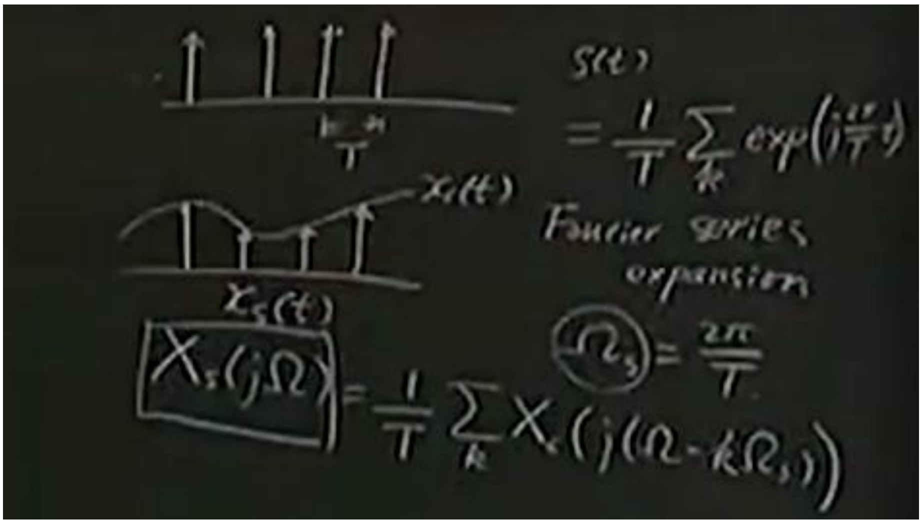
where $\Omega_s = 2\pi / T = 2\pi f_s$.

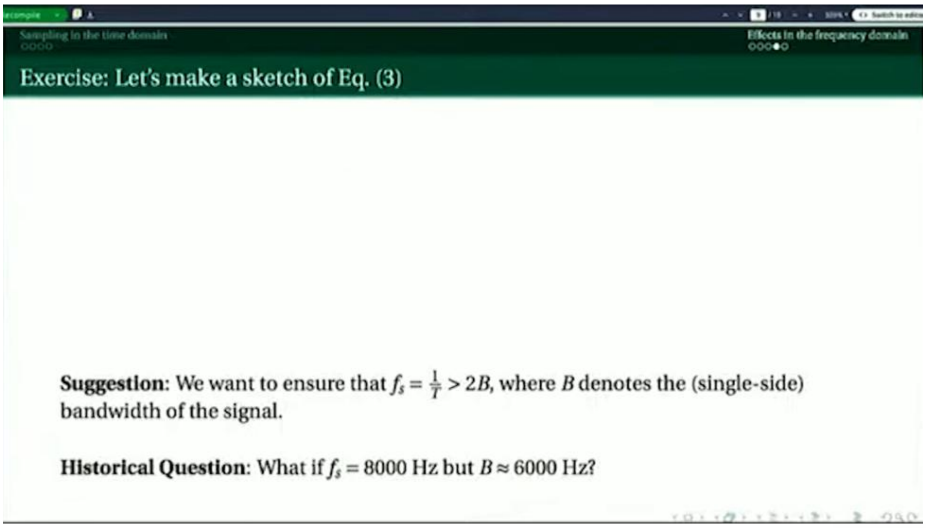
[1] In this course, if the range of summation is not specified, we mean summation from $-\infty$ to ∞.

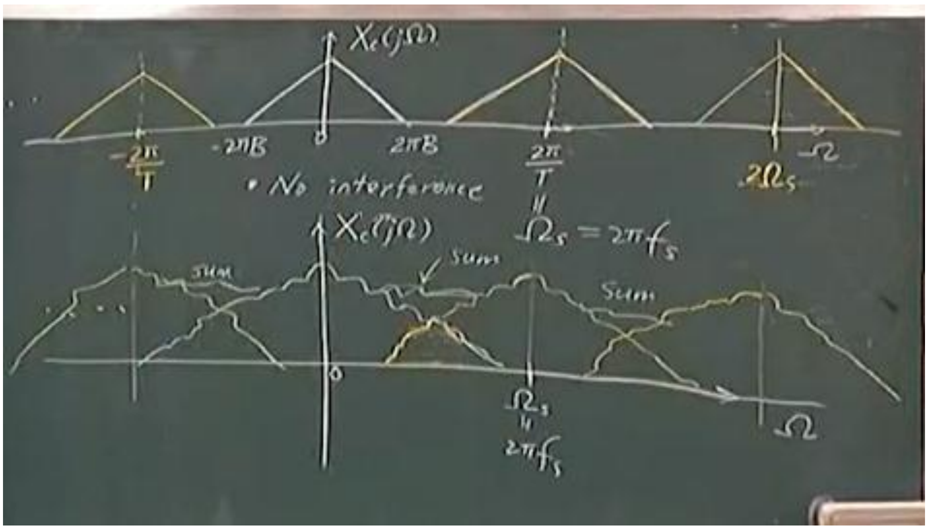


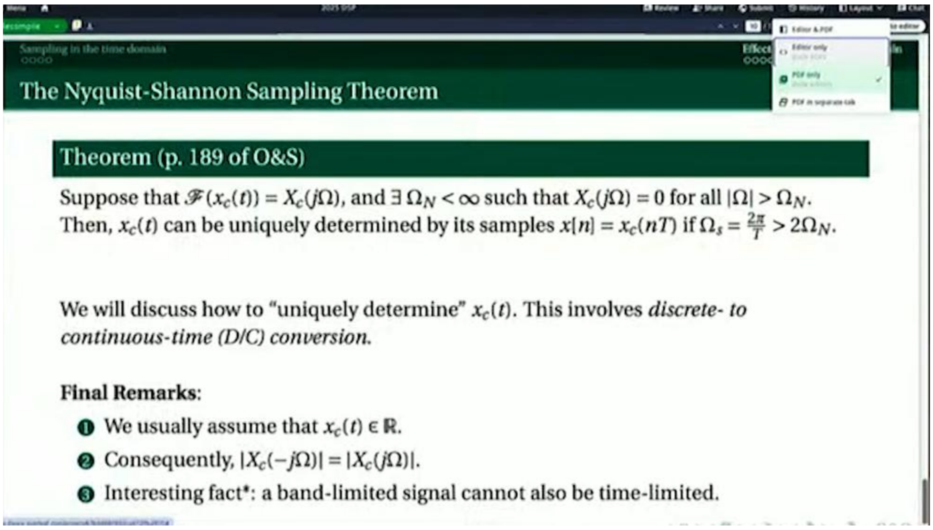


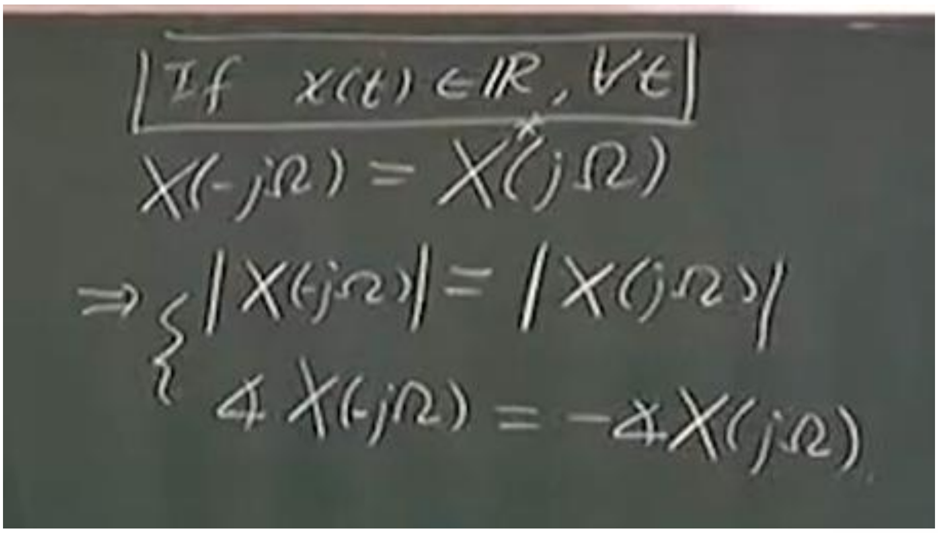

$S(t)$
t
$S(j\Omega)$
$\mathcal{F}(x(t)) = X(j\Omega)$
Shift Theorem
Ω
$\Omega_s = \frac{2\pi}{T}$
$X(j(\Omega-\Omega_0))$


$S(t)$
Fourier series
expansion
$x_c(t)$
$x_s(t)$
$\Omega_s = \frac{2\pi}{T}$
$X_s(j\Omega) = \frac{1}{T}\sum_k X_c(j(\Omega - k\Omega_s))$


Sampling in the time domain
Effects in the frequency domain
Exercise: Let's make a sketch of Eq. (3)
Suggestion: We want to ensure that $f_s = \frac{1}{T} > 2B$, where B denotes the (single-side) bandwidth of the signal.
Historical Question: What if $f_s = 8000$ Hz but $B \approx 6000$ Hz?


$X_c(j\Omega)$
$-\frac{2\pi}{T}$
$-2\pi B$
0
$2\pi B$
$\frac{2\pi}{T}$
$2\Omega_s$
Ω
• No interference
$X_c(j\Omega)$
$\Omega_s = 2\pi f_s$
sum
sum
Sum
0
Ω_s
$2\pi f_s$
Ω


Sampling in the time domain
The Nyquist-Shannon Sampling Theorem
Theorem (p. 189 of O&S)
Suppose that $\mathscr{F}(x_c(t)) = X_c(j\Omega)$, and $\exists\, \Omega_N < \infty$ such that $X_c(j\Omega) = 0$ for all $|\Omega| > \Omega_N$. Then, $x_c(t)$ can be uniquely determined by its samples $x[n] = x_c(nT)$ if $\Omega_s = \frac{2\pi}{T} > 2\Omega_N$.
We will discuss how to "uniquely determine" $x_c(t)$. This involves *discrete- to continuous-time (D/C) conversion.*
Final Remarks:
1. We usually assume that $x_c(t) \in \mathbb{R}$.
2. Consequently, $|X_c(-j\Omega)| = |X_c(j\Omega)|$.
3. Interesting fact*: a band-limited signal cannot also be time-limited.


If $x(t) \in \mathbb{R}, \forall t$
$X(-j\Omega) = X^*(j\Omega)$
$\Rightarrow \begin{cases} |X(-j\Omega)| = |X(j\Omega)| \\ \angle X(-j\Omega) = -\angle X(j\Omega) \end{cases}$